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Odvijanje preje z navitka - obravnava kinematičnih in dinamičnih lastnosti preje

Yarn unwinding from packages – a discussion on the kinematic and dynamic properties of yarn

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Obravnavamo odvijanje preje z navitkov, kar je ključnega pomena pri številnih tekstilnih procesih. Izpeljemo zelo splošen sistem diferencialnih enačb, ki opisujejo gibanje preje med odvijanjem.

Opisan je fizikalni pomen posameznih členov, ki nastopajo v enačbah, s posebnim poudarkom na navideznih silah v vrtečem se koordinatnem sistemu. Prikažemo tudi, kako lahko v kvazistacionarnem približku sistem enačb numerično rešimo in dobimo sliko preje v prostoru med odvijanjem.

(Ključne besede: gibanje preje, lastnosti kinematične, lastnosti dinamične, enačbe diferencialne)

We discuss yarn unwinding from packages, which is of chief importance in many textile processes. We derive a very general system of differential equations that describe the motion of the yarn during unwinding.

We discuss the physical meaning of individual terms in the equations with special emphasis on virtuals forces, which appear in rotating coordinate systems. We also show how the equations can be numerically solved in the quasistationary approximation in order to obtain an image of yarn in space during unwinding.

(Keywords: yarn motion, kinematic properties, dynamic properties, partial differential equations)

0 UVOD

Nihanja mehanske napetosti, do katerih prihaja med odvijanjem preje z navitka, povzročajo številne težave in lahko vplivajo na učinkovitost tekstilnega postopka in na kakovost končnega izdelka. Ta nihanja so še posebej opazna pri vzdolžnem odvijanju, pri katerem je navitek nameščen nepremično, preja pa se hitro odvija in teče stran od navitka v smeri njegove osi. Pomembno je, da poiščemo obliko navitka, pri kateri bo gibanje preje takšno, da bo napetost v preji majhna in čim bolj enakomerna.

Odvijanje preje bomo obravnavali s teoretičnega vidika. Izpeljali bomo sistem diferencialnih enačb, ki opisuje gibanje preje med odvijanjem. Enačbe bomo izpeljali z najmanjšim številom privzetkov, tako da bodo čim bolj splošne. Pri tem bomo dali velik poudarek fizikalnim

0 INTRODUCTION

Oscillations of tension in yarn, which appear when the yarn is unwinding from a package, cause many problems and can degrade the efficiency of the textile process and the quality of the end product. These oscillations are particularly strong in axial unwinding, where the package is stationary and the yarn is being withdrawn in the direction of package axis. It is thus important to find the optimum shape of the package for which the motion of the yarn will be such that the yarn tension will be small and as steady as possible.

The unwinding will be discussed from the theoretical point of view. We will derive a system of differential equations that provides a description of the yarn motion during the unwinding. We will derive these equations using a minimal set of assumptions, so that the resulting equations will retain their generality. An razlagam posameznih matematičnih izrazov in členov, ki se v njih pojavljajo: namen prispevka ni le razviti računski formalizem, temveč tudi bralcu omogočiti razumevanje bistvenih fizikalnih dejavnikov, ki lahko vplivajo na končni rezultat. Zato bomo bolj izdatno, kakor je sicer v navadi, spregovorili o opisu preje kot krivulji v prostoru in o njeni parametrizaciji, o podobnosti s hidrodinamičnim problemom toka tekočin (razlika med lokalnim in substancialnim odvodom), o uporabi Newtonovega zakona pri razsežnih telesih in o tem, kako opišemo napetostno stanje v enorazsežnih telesih. Na koncu bomo pokazali še, kako zapišemo enačbe v vrtečem se koordinatnem sistemu, opisali bomo navidezne "sistemske" sile, ki jih pri tem dobimo, ter izpeljali pogoj, ki ga dobimo zaradi privzetka nerazteznosti preje.

1 OPIS PREJE IN KINEMATIČNE LASTNOSTI

Pri obravnavi gibanja preje običajno zanemarimo prečno razsežnost preje. Z drugimi besedami, mislimo si, da je preja neskončno tanka in da jo zato lahko obravnavamo kot enorazsežni predmet. Takšni približki so v mehaniki pogosti: tudi kovinske žice, strune glasbenih inštrumentov, elastike, vlakna in podobne dolge, vendar tanke predmete, obravnavamo kot idealno tanka telesa. Napaka, ki jo s takšnih približkom storimo, je zanemarljivo majhna. Poleg tega privzamemo, da je preja neraztezna: to pomeni, da zanemarimo raztezke v preji, vendar pa kljub temu upoštevamo napetost v preji. Dokazano je bilo, da vodi ta približek le k majhni napaki pri običajno uporabljeni preji [1].

Enorazsežno telo opišemo kot krivuljo v prostoru. Najlaže jo podamo v parametrizirani obliki: vsako točko na krivulji podamo z njenim krajevnim vektorjem $\mathbf{r}(s)$, pri čemer je s parameter, s katerim oštevilčimo točke na krivulji. Pri opisu preje je najbolj naravna in pripravna parametrizacija z ločno dolžino, kar pomeni, da je s dolžina preje med izbrano točko $\mathbf{r}(s)$ in izbranim izhodiščem. Pri odvijanju preje z navitkov je najbolj primerna izbira izhodišča vodilo, skozi katerega prejo vlečemo z nespremenljivo hitrostjo (sl. 1).

Prejo odvijamo s hitrostjo V skozi vodilo O, ki je tudi izhodišče koordinatnega sistema. Točka **Dv** je točka dviga, to je točka, v kateri preja zapusti

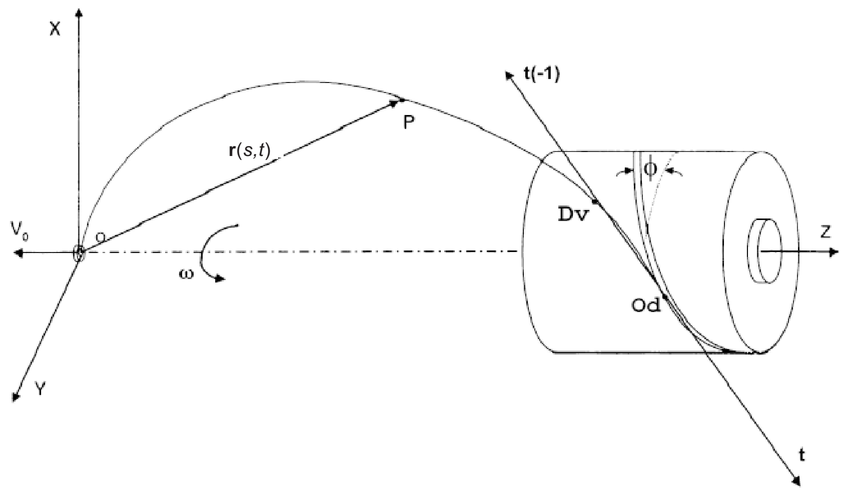
emphasis will be given to the physical interpretations of the mathematical expressions and of the various terms that appear in them: the purpose of this paper is not only to develop a formalism for the calculations, but also to lead to an understanding of the key physical elements that have a direct influence on the final result. For this reason we will amply describe details that are usually neglected, such as the description of the yarn as a curve in space and its parametrisation, the similarity with the hydrodynamical problem of liquid flow (the difference between the local and the substantial derivative), the use of Newton's law for a description of the extended bodies and the description of the elastic state of one-dimensional objects. We will show how the transition to the rotating cylindrical coordinate system is accomplished and how this leads to virtual "system" forces. The condition resulting from the non-extensibility of the yarn will also be given.

1 DESCRIPTION OF THE YARN AND THE KINEMATIC PROPERTIES

The lateral dimensions of the yarn are usually neglected in descriptions of yarn motion. In other words, we consider the yarn to be infinitely thin, so that it can be described as a one-dimensional object. Such approximations are common in mechanics, they are used for objects such as metal wires, the strings of musical instruments, fibers and other long, but thin objects, that can be considered as ideally thin. This approximation leads to a small error that can be safely neglected. We furthermore assume that the yarn is not extensible: by this we mean that any elongation is neglectable; however, we do take into account the tension in the yarn. It has been shown that this assumption leads to only a small error with commonly used yarns [1].

A one-dimensional body can be mathematically described as a space curve. It can be most conveniently given in parametric form: each point on the curve is described by its coordinates (radius vector) $\mathbf{r}(s)$, where s is some parameter used to number the points on the curve. For yarns the most natural and useful parametrisation is given by its arc length. Parameter s is then the length of yarn between the chosen point $\mathbf{r}(s)$ and the origin. A very convenient choice of the origin for yarn-unwinding problems is the guide through which the yarn is being pulled away with constant velocity (Fig. 1).

The yarn is being withdrawn with unwinding speed V through a guide O, which also serves as the origin of the coordinate system. Point **Dv** is the lift-off point, i.e., the point where the yarn leaves the surface



Sl. 1. *Odvijanje preje z valjastega navitka*
Fig. 1. *Yarn unwinding from a cylindrical package*

površino navitka in naprej ustvarja balon. Točka **Od** je točka, kjer se preja začne odvijati in drseti po navitku. Kot ϕ je kot navijanja preje na valjasti navitek. Vektor **k** je tangentni vektor na prejo v točki odvijanja.

Zanimalo nas bo gibanje preje, torej časovno spreminjanje lege krivulje **r**(*s*) v prostoru. Zato uvedemo še dodaten parameter *t*, ki podaja trenutek, ob katerem ima krivulja obliko **r**(*s*,*t*=*konst*). Rečemo lahko tudi, da je funkcija **r**(*s*,*t*) pri izbranem stalnem času *t* "trenutna slika preje v prostoru", kakor bi jo posneli s fotoaparatom. Časovni potek odvijanja preje zato podamo kot dvoparametrično vektorsko funkcijo **r**(*s*,*t*).

Pri kinematiki odvijajoče se preje naletimo na podobno težavo kakor pri hidrodinamičnem problemu toka tekočin [2]. Problema sta si podobna v tem, da imamo tudi tukaj opravka s prejo, ki "teče" proti vodilu vzdolž svoje "struge", ki jo v nekem trenutku t_o podaja funkcija **r**($s,t=t_o$). Dejansko sta pri preji sočasno prisotni dve različni gibanji: "tok" preje, ki jo vlečemo proti vodilu (to gibanje je v vsaki točki tangentno na krivuljo **r**($s,t=t_o$)), ter spreminjanje "struge" same, zaradi odvijanja preje z navitka.

Lega preje je, kakor rečeno, odvisna tako od ločne doline *s* kakor od časa *t*: **r**=**r**(*s*,*t*). Iz zaporednih opazovanj lege pri eni in isti ločni dolžini dobimo *lokalni časovni odvod*, ki ga označimo z ∂**r**/∂t. Ta odvod pa ni enak hitrosti preje! Enak je spreminjanju "struge" preje, ne upošteva pa dejstva, da preja "teče po strugi" s hitrostjo odvijanja *V*.

of the packages to form the balloon. Point **Od** is the unwinding point, where the yarn starts to slide on the surface of the package. Angle ϕ is the angle of winding on the package. Vector **k** is the tangent vector on the yarn at the unwinding point.

We are interested in yarn motion, i.e., the time variation of the position **r**(*s*) of the yarn in space. We therefore introduce an additional parameter *t*, which gives us a moment in time when the form of the yarn is **r**(*s*,*t*=*const*). In other words, the function **r**(*s*,*t*) at fixed time *t* is a "snapshot of the yarn in space" as taken by a camera. The process of yarn unwinding is therefore described using a two-parameter vector function **r**(*s*,*t*).

In a kinematic description of yarn unwinding we face a similar problem to that in the hydrodynamic problem of liquid flow [2]. Both problems are similar in that the yarn also "flows" in the direction of the guide along its "riverbed", which at time t_o is given by the function **r**($s,t=t_o$). In fact there are two simultaneous motions of the yarn: the "flow" of the yarn being pulled through the guide (this motion is at every point tangent to the curve **r**($s,t=t_o$)) and the time variation of the "riverbed" itself due to the unwinding of the yarn from the package.

The position of a point on the yarn depends on two parameters: the arc length *s* and the time *t*: **r**=**r**(*s*,*t*). If we observe the position of the yarn at a fixed arc length we obtain a *local time derivative*, denoted by ∂**r**/∂t. This derivative, however, is not equal to the yarn velocity. Instead, it is equal to the velocity of the changes of the "riverbed", but it does not take into account that the yarn "flows within the riverbed" with the unwinding speed *V*.

Če hočemo izmeriti hitrost izbrane točke preje, ne smemo preje opazovati pri nespremenljivem s, temveč se moramo vzdolž "toka" gibati skupaj s prejo. Gibanje odseka preje opiše funkcija $\mathbf{r}(s(t),t)$, pri čemer je $s(t)$ ločna dolžina obravnavanega odseka preje ob času t. Lega tega odseka je odvisna samo od časa. Hitrost dobimo kot totalni odvod, $\mathbf{v}=d(\mathbf{r}(s(t),t)/\mathrm{d}t$. V hidrodinamiki je takšen odvod znan kot *substancialni odvod*. Z *lokalnim odvodom* $\partial\mathbf{r}/\partial$t ga povežemo z uporabo pravil diferencialnega računa:

If one wants to measure the velocity of a chosen point on the yarn, one should not observe the yarn at fixed s, but should instead follow the "flow" of the yarn. The motion of a short segment of the yarn is described by the function $\mathbf{r}(s(t),t)$, where $s(t)$ is the arc-length of the segment at time t. The position of the segment is a function of time only. The velocity can be obtained using a total derivative, $\mathbf{v}=d(\mathbf{r}(s(t),t))/dt$. Such a derivative is known in hydrodynamics as a *substantial derivative*. It can be related to the *local derivative*$\partial\mathbf{r}/\partial$t using the chain rule of calculus:

$$\frac{\mathrm{d}\mathbf{r}(s(t),t)}{\mathrm{d}t} = \frac{\partial\mathbf{r}}{\partial t} + \frac{\partial\mathbf{r}}{\partial s}\frac{\mathrm{d}s}{\mathrm{d}t} \quad (1).$$

Sedaj upoštevamo privzetek, da je preja neraztezna. Če prejo odvijamo z odvijalno hitrostjo V, ki se s časom ne spreminja, potem za vsak kratek odsek preje velja $s(t)=s_o-Vt$. Predznak minus dobimo zato, ker prejo vlečemo v smeri vodila, zato se ločna dolžina $s(t)$ izbranega odseka zmanjšuje s časom linearno proti nič. V poljubni točki na preji tedaj velja $ds/dt=-V$. Zapišemo torej:

We now take into account that the yarn was assumed inextensible. If the yarn is withdrawn with an unwinding speed V that is constant with time, then for any short segment of the yarn we have $s(t)=s_o-Vt$. We obtain a minus sign because the yarn is being pulled in the direction of the guide, so that the arc-length $s(t)$ to a given segment decreases with time linearly toward zero. Therefore we have $ds/dt=-V$ at any point of the yarn. We can then write:

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\partial\mathbf{r}}{\partial t} - V\frac{\partial\mathbf{r}}{\partial s} \quad (2).$$

Ker je parametrizacija z ločno dolžino naravna parametrizacija, je $\partial\mathbf{r}/\partial$s enotski tangentni vektor $\boldsymbol{k}$ na prejo v dani točki. Končna enačba za hitrost se torej glasi:

As arc-length parametrisation is a natural parametrisation of a curve, the derivative $\partial\mathbf{r}/\partial$s is equal to the unit tangent vector $\mathbf{k}$ to the yarn at a given point. The final expression for the velocity of a segment is therefore:

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\partial\mathbf{r}}{\partial t} - V\mathbf{k} \quad (3).$$

Prvi člen opisuje "spreminjanje struge", drugi člen pa tangentni "tok" preje proti vodilu. Na tem mestu moramo poudariti, da izraz (3) nikakor ne pomeni, da je tangentna komponenta hitrosti enaka $-V$ v vseh točkah preje, saj lahko tudi člen $\partial\mathbf{r}/\partial$t vsebuje komponento v tej smeri. Medtem ko je tangenta komponenta hitrosti po definiciji naloge enaka V pri vodilu, je tangentna komponenta zagotovo enaka nič v točki odvijanja **Od**, kjer se preja ravno začne premikati.

Vpeljemo lahko abstraktni operator totalnega časovnega odvoda D, ki sledi gibanju točke na preji:

The first term describes the changing "riverbed" and the second term gives the tangential "flow" of the yarn along the riverbed in the direction of the guide. At this point we should emphasize that the expression (3) in no way implies that the tangential component of the velocity is equal to $-V$ at all points on the yarn, because the term $\partial\mathbf{r}/\partial$t can also contain a component along this direction. Indeed, while the tangential component of the velocity is from the definition of the problem equal to V at the guide, it is clearly equal to zero at the unwinding point **Od** where the yarn just starts to move.

We can introduce an abstract total time derivative operator D, which follows the motion of a point along the yarn:

$$D = \frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} - V\frac{\partial}{\partial s} \quad (4).$$

Če operator D uporabimo na krajevnem vektorju $\mathbf{r}$, dobimo izraz (2). Hitrost gibanja odseka

By applying the operator D on a radius vector $\mathbf{r}$, we obtain expression 1. The velocity of the yarn segment preje je torej $\mathbf{v}=D\mathbf{r}$. Pričakujemo torej, da bomo dobili pospešek odseka preje, če operator D uporabimo dvakrat na krajevnem vektorju: $\mathbf{a}=D^2\mathbf{r}$. Dobimo:

is therefore $\mathbf{v}=D\mathbf{r}$. We can expect that the acceleration of a yarn segment is given by applying the operator D twice on the radius vector: $\mathbf{a}=D^2\mathbf{r}$. We obtain:

$$D^2\mathbf{r} = \left(\frac{\partial}{\partial t} - V\frac{\partial}{\partial s}\right)^2 \mathbf{r} = \left(\frac{\partial^2}{\partial t^2} - 2V\frac{\partial^2}{\partial s \partial t} + V^2\frac{\partial^2}{\partial s^2}\right)\mathbf{r} \quad (5),$$

torej

or

$$\mathbf{a} = \frac{\partial^2 \mathbf{r}}{\partial t^2} - 2V\frac{\partial \mathbf{k}}{\partial t} + V^2\frac{\partial \mathbf{k}}{\partial s} \quad (6).$$

Ta rezultat lahko preverimo tudi z neposrednim izračunom pospeška brez uporabe operatorja D.

We can ascertain that this is in fact the correct expression for acceleration by a direct calculation without using the operator D.

2 DINAMIKA: NEWTONOV ZAKON ZA ODSEK PREJE

2 DYNAMICS: NEWTON'S LAW FOR A YARN SEGMENT

Gibanje preje bomo opisali v inercialnem opazovalnem sistemu, v katerem velja Newtonov zakon $\mathbf{F}=m\mathbf{a}$, kjer so $\mathbf{F}$ sila na telo, $\mathbf{a}$ pospešek, m pa masa telesa. Newtonov zakon je zapisan v obliki, ki je uporabna za obravnavo gibanja snovnih delcev (na primer atomov), togih teles (krogel, planetov itn.), ter za opis gibanja *težišča* deformljivih teles kot celote. Pri preji, ki je deformljivo telo, nas ne zanima, kako se giblje preja kot celota, temveč kako se spreminja oblika preje same. Zato prejo v mislih razrežemo na (infinitezimalno) kratke odseke dolžine δs in uporabimo Newtonov zakon za vsak odsek posebej.

Oglejmo si najprej, katere sile delujejo na naš sistem, torej na kratek odsek preje, katerega eno krajišče je v točki $\mathbf{r}(s)$, drugo krajišče pa v točki $\mathbf{r}(s+\delta s)$. Očitno je odsek izpostavljen sili težnosti $\mathbf{F}_t=m\mathbf{g}$ (m je masa odseka preje, $\mathbf{g}$ pa težnostni pospešek). Izkaže se, da je sila težnosti zanemarljiva v primerjavi z drugimi silami, zato jo zanemarimo [3]. Če se odsek premika po zraku, nanj deluje tudi sila zračnega upora $\mathbf{F}_{zr}$:

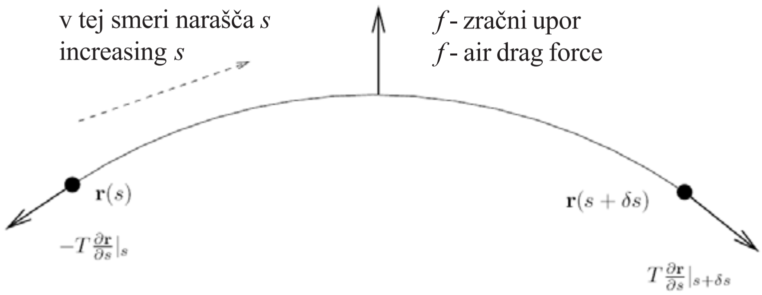
We will describe yarn motion in an inertial observation frame where Newton's law $\mathbf{F}=m\mathbf{a}$ is valid. Here, $\mathbf{F}$ is the force acting on the body, $\mathbf{a}$ is the acceleration and m is the mass of the body. This form of Newton's law is appropriate for describing the motion of material particles (such as atoms), rigid bodies (balls, planets etc.) and for describing the motion of the *center of mass* of deformable bodies. The yarn is deformable and we are not interested in how it moves as a whole. We are interested instead in the changes of the form of the yarn. We therefore divide the yarn into (infinitesimally) short segments of length δs and apply Newton's law to each segment individually.

First we need to determine which forces act on our system, i.e., on the short segment of yarn whose one extremity is $\mathbf{r}(s)$ and other extremity is $\mathbf{r}(s+\delta s)$. The force of gravity $\mathbf{F}_t=m\mathbf{g}$ (m is the mass of the segment and $\mathbf{g}$ the gravitational acceleration) acts on the segment. It turns out that the effect of gravitation can usually be neglected in comparison to the other forces [3]. If the segment is moving through the air, there is also a contribution from the air drag force $\mathbf{F}_{zr}$:

$$F = \frac{1}{2}c_u \rho v_n^2 S \quad (7),$$

kjer so: c_u koeficient zračnega upora, ρ gostota zraka, $\mathbf{v}_n$ pravokotna komponenta hitrosti, S pa čelni prerez odseka preje ([4] in [3]). Reynoldsevo število je višje od 1000 v tistih delih preje, kjer je hitrost največja, in nižje drugod. Kljub temu uporabimo kvadratni zakon zračnega upora na celotni dolžini preje, saj velja v tistih delih preje, kjer je učinek zračnega upora največji.

where c_u is the coefficient of air drag, ρ is the density of the air, v_n is the normal component of the velocity and S is the frontal area of the yarn segment ([4] and [3]). The Reynolds number is higher than 1000 on the parts of the yarn with the highest velocity and smaller elsewhere. Nevertheless, we use the law of quadratic air drag on the entire length of the yarn because it is valid on those parts of the yarn where it has the largest effect.



Sl. 2. *Sila na odsek preje*
Fig. 2. *Forces acting on a yarn segment*

Na kratek odsek preje pa neposredno delujeta tudi preostala kosa preje na obeh straneh obravnavanega odseka (sl. 2), zato na vsako krajišče deluje neka sila. Ti sili sta posledica notranjega napetostnega stanja zaradi natezne obremenjenosti preje, podobno kakor pri napeti elastiki.

Na odsek preje delujejo sile zaradi napetosti in sila zračnega upora.

V trorazsežnih telesih (kontinuih) napetostno stanje opišemo z napetostnim tenzorjem, v enorazsežnem telesu, kakršna je preja, pa zadostuje skalarna količina, imenovana napetost *T*. Ta pove, kakšna sila deluje na rob enorazsežnega telesa zaradi deformacij in ima enoto sile [*N*]. Definiramo jo z enačbo:

In addition to these obvious forces, there are also forces imparted on the segment by the remaining yarn on each side of the segment, see Fig. 2, so that there is an additional force on each extremity of the segment. These two forces are a consequence of the internal elastic state due to elastic strain on the yarn, similar to the case of a stretched elastic band.

Forces of tension and air friction force act on a short yarn segment.

In three-dimensional continuum bodies the stress state is given by the stress tensor, whereas in one-dimensional bodies such as the yarn, a single scalar quantity is sufficient. This quantity is called the tension, and it is denoted by *T*. The tension is numerically equal to the force that acts on an extremity of a one-dimensional body due to deformations and it has the same unit as force, i.e., Newton [N]. It is defined with the equation:

$$\mathbf{F} = T\mathbf{k} \tag{8}$$

kjer sta **F** sila na rob obseka preje, vektor **k** pa tangentni vektor na prejo v točki prijemališča sile **F**, torej v robni točki. Sila na rob preje v točki **r**(s) je:

where **F** is the force acting on the extremity of the segment, vector **k** is a tangent vector on the yarn at the point of the application of the force **F**, i.e., at the extremity. The force on the extremity at **r**(s) is therefore:

$$-T(s)\mathbf{k}(s) \tag{9}$$

sila na drugi rob v točki **r**($s+\delta s$) pa:

and the force on the other extremity at **r**($s+\delta s$) is

$$T(s+\delta s)\mathbf{k}(s+\delta s) \tag{10}$$

Drugi Newtonov zakon za odsek preje zato zapišemo kot:

Newton's second law for a segment of yarn can be expressed as

$$m\mathbf{a} = T(s+\delta s)\mathbf{k}(s+\delta s) - T(s)\mathbf{k}(s) + \mathbf{F}_{zr} \tag{11}$$

Masa odseka je $m=\rho\delta s$, kjer je ρ linearna gostota (masa na enoto dolžine preje), silo F_{zr} pa

The mass of the segment is $m=\rho\delta s$, where ρ is the mass of yarn per unit length. The force $\mathbf{F}_{zr}$ can zapišemo kot $\mathbf{F}_{zr}=\mathbf{f}_{zr}\delta s$, kjer je $\mathbf{f}_{zr}$ linearna gostota sile zračnega upora (torej sila zračnega upora na enoto dolžine preje). Zato enačbo (11) delimo z δs in naredimo limiti proti infinitezimalno kratki dolžini odseka, $\delta s \to 0$:

be written as $\mathbf{F}_{zr}=\mathbf{f}_{zr}\delta s$, where $\mathbf{f}_{zr}$ is the linear density of the air drag force (the air drag force per unit length). By dividing Equation (11) by δs and going to the limit of an infinitesimally short segment length, $\delta s \to 0$, we obtain:

$$\rho\mathbf{a} = \lim_{\delta s\to 0}\frac{T(s+\delta s)\mathbf{k}(s+\delta s)-T(s)\mathbf{k}(s)}{\delta s}+\mathbf{f}_{\mathrm{zr}} \qquad (12).$$

Limita v zgornjem izrazu je po definiciji odvod funkcije $T(s)\mathbf{k}(s)$ po ločni dolžini s. Končni rezultat, torej gibalna enačba za infinitezimalno kratek odsek preje, je:

The limit in this expression is, by definition, the arc-length derivative of the function $T(s)\mathbf{k}(s)$. The final result, the equation of motion of an infinitesimally short segment of yarn, is then

$$\rho\mathbf{a}(s) = \frac{\partial}{\partial s}\Big(T(s)\mathbf{k}(s)\Big)+\mathbf{f}_{\mathrm{zr}}(s) \qquad (13).$$

Če uporabimo rezultat za pospešek, (6), jo lahko zapišemo tudi v obliki:

Using the expression (6) for the acceleration, the equation of motion can also be put in the form

$$\rho\left(\frac{\partial^2\mathbf{r}}{\partial t^2}-2V\frac{\partial\mathbf{k}}{\partial t}+V^2\frac{\partial\mathbf{k}}{\partial s}\right)=\frac{\partial}{\partial s}\Big(T\mathbf{k}\Big)+\mathbf{f}_{\mathrm{zr}} \qquad (14).$$

3 PREHOD V VRTEČI SE VALJNI KOORDINATNI SISTEM

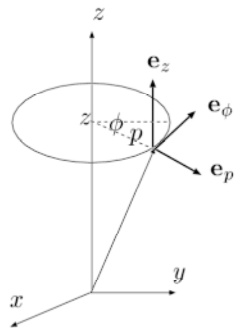
Pri odvijanju preje z navitka ustvarja preja "balon" (slika 1): preja se z veliko kotno hitrostjo vrti okoli osi Z in v eni periodi oriše rotacijsko telo z enim ali več "trebuhi". Oblika rotacijskega telesa - balona, se v času ene periode vrtenja spremeni le malo. Celotno gibanje krivulje $\boldsymbol{r}(s)$ lahko torej razstavimo na dve gibanji z različnima značilnima časoma. Prvo gibanje je vrtenje krivulje okoli osi Z in ima kratek značilni čas (reda $2\pi/\omega$, kjer je ω kotna hitrost: to je čas, v katerem se odvije en ovoj niti). Drugo gibanje je spreminjanje oblike balona in ima dolg značilni čas (ta je velikostnega reda časa, v katerem se odvije ena plast).

Takšen razcep je primeren predvsem za navitke (oziroma plasti), pri katerih je število ovojev veliko, torej predvsem za natančno navite navitke. Recimo, da ima ena plast navitka okoli 100 ovojev niti. Tedaj se oba značilna časa razlikujeta za dva velikostna reda. Imamo torej opravka z dvema gibanjema na zelo različnih časovnih merilih, zato je smiselno, da takšen razcep izrecno upoštevamo v naših enačbah. Pri navitkih z manjšim številom ovojev je razcep manj uporaben in naloge se je bolje lotiti z neposrednim numeričnim reševanjem enačbe (14), kar pa je izjemno težko.

3 TRANSITION TO A ROTATING CYLINDRICAL COORDINATE SYSTEM

The unwinding yarn forms a "balloon" (Fig. 1): the yarn rotates with a high angular velocity around the Z axis and with one turn it defines the contour of a rotational body with one or several balloons. The shape of the rotational body – the balloon – changes only a little in one period of the motion. The motion of the curve $\mathbf{r}(s)$ can therefore be decomposed into two separate motions with two different characteristic times. The first motion is the rotation of the rigid curve around the Z axis. It has a short characteristic time $2\pi/\omega$, where ω is the angular velocity. In this time one loop of the yarn is unwound. The second motion corresponds to the time-varying form of the balloon. This has a long characteristic time, of the order of the time in which one layer of the yarn is unwound from the package.

Such a decomposition makes sense only for packages (or layers) where the number of loops in a layer is high, i.e., for precision-wound packages. Let one layer have 100 loops of yarn. Then both characteristic times differ by two orders of magnitude and it is beneficial to take this decomposition explicitly into account in our equation of motion. In packages with a smaller number of loops such a decomposition is less useful and the problem is best approached by directly numerically solving the Equation (14); however, this is a very difficult task.



Sl. 3. *Valjni koordinatni sistem*
Fig. 3. *Cylindrical coordinate system*

Vsaka točka ima lastno trojico osnovnih vektorjev $\mathbf{e}_p$,$\mathbf{e}_\phi$,$\mathbf{e}_z$.

Najprej se iz kartezičnega koordinatnega sistema preselimo v valjnega. Ta je bolj primeren za probleme, v katerih obstaja simetrijska os. Počasi spreminjajoče se rotacijsko telo, balon, dejansko ima takšno simetrijsko os, zato bo obravnava *hitrega* dela gibanja (vrtenja) v takšnem koordinatnem sistemu lažja. V tem koordinatnem sistemu točko opišemo s koordinatami p (oddaljenost točke od osi z), polarnim kotom ϕ in višino točke z, kakor je prikazano na sliki 3. Spremembo zapišemo z naslednjimi enačbami:

To each point there corresponds a different triplet of basis vectors $\mathbf{e}_p$, $\mathbf{e}_\phi$, $\mathbf{e}_z$.

We first affected the change from a Cartesian to a cylindrical coordinate system. A cylindrical coordinate system is more appropriate for problems that possess a symmetry axis. The slowly deforming rotational body, the balloon, does have such an axis, therefore the *fast* motion (rotation) can be handled more easily in this coordinate system. In the cylindrical system the position of a point is given by coordinates p (the distance from the Z axis), the polar angle ϕ and the height z, as shown in Fig. 3. The transformation can be expressed using the following equations:

$$p = \sqrt{x^2 + y^2}, \quad \phi = \arctan(y/x), \quad z = z \tag{15}$$

Krajevni vektor zapišemo kot:

The radius vector can be put in the form:

$$\mathbf{r} = p\mathbf{e}_p + z\mathbf{e}_z \tag{16}$$

Vektorji $\mathbf{e}_p$, $\mathbf{e}_\phi$ in $\mathbf{e}_z$ so osnovni vektorji v točki $\mathbf{r}$. Paziti moramo na dejstvo, da ima vsaka točka svojo trojico osnovnih vektorjev: vektorja $\mathbf{e}_p$ in $\mathbf{e}_\phi$ sta odvisna od kota ϕ, kar je razvidno s slike 3. Vektor $\mathbf{e}_z$ pa je enak v vseh točkah. Odvisnosti od parametrov s in t zapišemo še izrecno v obliki:

Vectors $\mathbf{e}_p$, $\mathbf{e}_\phi$ and $\mathbf{e}_z$ are the basis vectors proper to point $\mathbf{r}$. We have to pay attention to the fact that each point has its proper triplet of basis vectors: vectors $\mathbf{e}_p$ and $\mathbf{e}_\phi$ depend on the polar angle ϕ, as shown in Fig. 3; vector $\mathbf{e}_z$ is the same in all points. For clarity we can write the dependence of the different terms in the expression for the radius vector on parameters s and t explicitly:

$$\mathbf{r}(s,t) = p(s,t)\mathbf{e}_p(\phi(s,t)) + z(s,t)\mathbf{e}_z \tag{17}$$

V nadaljevanju bomo potrebovali še razmerji:

Later we will need the following two relations

$$\frac{\partial \mathbf{e}_p}{\partial \phi} = \mathbf{e}_\phi, \qquad \frac{\partial \mathbf{e}_\phi}{\partial \phi} = -\mathbf{e}_p, \tag{18}$$

ki sta osnovna lastnost valjnih koordinatnih sistemov in ju dobimo, če naredimo infinitezimalno

which are a basic property of cylindrical coordinate systems and can be obtained by performing an infinitesimal zavrtitev koordinatnega sistema okoli osi z za kot $\delta\phi$. Z njima lahko tudi dokažemo, da velja:

rotation of the coordinate system around the z axis. These relations can be used to derive two useful formulas:

$$\dot{\mathbf{e}_p} = \frac{\partial \mathbf{e}_p}{\partial \phi}\frac{\partial \phi}{\partial t} = \dot{\phi}\mathbf{e}_\phi \quad (19)$$

in

and

$$\dot{\mathbf{e}_\phi} = \frac{\partial \mathbf{e}_\phi}{\partial \phi}\frac{\partial \phi}{\partial t} = -\dot{\phi}\mathbf{e}_p \quad (20).$$

Tu pika nad simbolom označuje parcialni odvod po času, ∂/∂t.

Hitro gibanje je vrtenje preje okoli osi z kot celote. To pomeni, da se pri tem gibanju polarni koti vseh točk spremenijo za enak kot na enoto časa. To zapišemo kot:

Here the dot above a symbol denotes a partial derivative with respect to time, ∂/∂t.

The fast motion corresponds to the rotation of the yarn as a whole around the z axis. In other words, the polar angle of each point on the yarn changes by the same amount per unit time. This can be expressed as:

$$\phi(s,t) = \omega t + \theta(s,t) \quad (21).$$

Tu smo predpostavili, da je kotna hitrost vrtenja ω, stalna. Poudariti moramo, da je kot $\phi(s,t)$ polarni kot točke v inercialnem valjnem sistemu (p, ϕ,z), kot $\theta(s,t)$ pa polarni kot točke znotraj vrtečega se koordinatnega sistema (p,ϕ,z). Če točka v vrtečem se sistemu "miruje" (θ je stalen), potem se točka v inercialnem sistemu enakomerno vrti okoli osi z s kotno hitrostjo ω. Če se kotna hitrost s časom spreminja, moramo zgornjo enačbo popraviti in dobimo:

Here we assumed that the angular velocity of rotation, ω, is constant. We point out that the angle $\phi(s,t)$ is the polar angle of the point in the inertial cylindrical system (p, ϕ,z), while the point $\theta(s,t)$ is the polar angle of the point in the rotating cylindrical system (p,θ,z). If a point is "motionless" in the rotating frame (i.e., if θ is constant), then such a point rotates in the inertial system with a constant angular velocity ω around the z axis. If the angular velocity varies, the previous equation has to be modified to read:

$$\phi(s,t) = \int_{t_0}^{t} \omega(\tau)\mathrm{d}\tau + \theta(s,t) \quad (22).$$

Hitrost $\boldsymbol{v}=D\mathbf{r}$ in pospešek $\boldsymbol{a}=D^2\mathbf{r}$ izračunamo z neposrednim odvajanjem krajevnega vektorja (17). Podroben izračun lahko bralec najde v viru [5], tu pa bomo navedli le končni rezultat.

Uvedemo vektor relativne hitrosti $v_{\mathrm{rel}} = \dot{r}\mathrm{e}_{\mathrm{r}} + r\dot{\theta}\mathrm{e}_\phi + \dot{z}\mathrm{e}_{\mathrm{z}}$: to je hitrost počasnega (relativnega) gibanja v hitro vrtečem se koordinatnem sistemu. Hitrost točke na preji lahko potem zapišemo kot:

Velocity $\mathbf{v}=D\mathbf{r}$ and acceleration $\mathbf{a}=D^2\mathbf{r}$ have to be calculated by explicit differentiation of the expression for the radius vector (17). A detailed account of this calculation can be found in Ref. [5], here we will only provide the final result.

We introduce the relative velocity vector $v_{\mathrm{rel}} = \dot{r}\mathrm{e}_{\mathrm{r}} + r\dot{\theta}\mathrm{e}_\phi + \dot{z}\mathrm{e}_{\mathrm{z}}$: this is the velocity of the slow (relative) movement within the rapidly rotating coordinate system. The velocity of a point on the yarn can then be expressed as:

$$\mathbf{v} = D\mathbf{r} = \mathbf{v}_{\mathrm{rel}} + \boldsymbol{\omega} \times \mathbf{r} - V\mathbf{k} \quad (23).$$

Razvidno je, od kod izvirajo posamezni členi:
1. člen $\mathbf{v}_{\mathrm{rel}}$ je, kakor je rečeno, relativna hitrost gibanja točke P v vrtečem se koordinatnem sistemu;
2. člen ωx$\mathbf{r}$ je hitrost kroženja točke P okoli osi Z s trenutno kotno hitrostjo $\omega(t)$;
3. člen -$V\mathbf{k}$ je hitrost, ki jo ima točka P na preji zaradi tega, ker prejo vlečemo skozi vodilo.

The origin of the different terms is fairly clear:
1. the term $\mathbf{v}_{\mathrm{rel}}$ is the relative velocity of the point P in the rotating coordinate system;
2. the term $\omega\times\mathbf{r}$ is the velocity of the circular motion of point P around the Z axis with momentary angular velocity $\omega(t)$;
3. the term -$V\mathbf{k}$ is the velocity of the point P because the yarn is withdrawn through the guide.

Če uvedemo še relativni pospešek $a_{rel} = (\ddot{r} - r\dot{\theta}^2)\mathrm{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathrm{e}_\theta + \ddot{z}\mathrm{e}_z$, lahko z njim zapišemo pospešek kot [5]:

Introducing the relative acceleration $a_{rel} = (\ddot{r} - r\dot{\theta}^2)\mathrm{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathrm{e}_\theta + \ddot{z}\mathrm{e}_z$ we can write the acceleration in the form [5]:

$$\mathbf{a} = \mathbf{a}_{\mathrm{rel}} + 2\boldsymbol{\omega}\times\mathbf{v}_{\mathrm{rel}} - 2V\boldsymbol{\omega}\times\mathbf{r}' + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) + \dot{\boldsymbol{\omega}}\times\mathbf{r} - 2V\mathbf{v}_{\mathrm{rel}}' + V^2\mathbf{r}'' \qquad (24).$$

Tu znak ′ pomeni parcialni odvod po ločni dolžini s, torej $\partial/\partial s$. Uvedemo lahko operator totalnega časovnega odvoda $\mathcal{D}$, ki sledi gibanju izbrane točke znotraj vrtečega se koordinatnega sistema:

Here the suffix ′ denotes the partial derivative with respect to the arc length s, i.e. $\partial/\partial s$. We can introduce a formal operator of the total time derivative $\mathcal{D}$, which follows the motion of the chosen point within the rotating coordinate system:

$$\mathfrak{D} = \frac{\partial}{\partial t}\Big|_{(p,\theta,z)} - V\frac{\partial}{\partial s} \qquad (25).$$

To, da operator sledi gibanju izbrane točke znotraj vrtečega se koordinatnega sistema, pomeni, da pri odvajanju izraza (21) ali (22) po času zanemarimo člen z ω. Tako dosežemo, da opravek odvajanja po času deluje *znotraj* vrtečega se koordinatnega sistema (p, θ, z), namesto v inercialnem sistemu (p,ϕ,z), na kar smo opomnili z oznako (p, θ, z) pri operatorju za odvajanje v izrazu (25). Tako dobimo na primer:

By requiring that the operator follows the motion within the rotating coordinate system we mean that when calculating the time derivative of the expressions (21) or (22) we should not take into account the term in ω. In this way we ensure that the time derivation applies *within* the rotating coordinate system (p,θ,z), instead of in the inertial system (p,ϕ,z). As a reminder we write the subscript (p, θ, z) after the time derivative in the expression (25). For example, we thus obtain:

$$\begin{aligned} \mathfrak{D}\mathbf{r} &= \mathbf{v}_{\mathrm{rel}} - V\mathbf{k}, \\ \mathfrak{D}^2\mathbf{r} &= \mathbf{a}_{\mathrm{rel}} - 2V\dot{\mathbf{k}} + V^2\mathbf{r}''. \end{aligned} \qquad (26).$$

Dobljena izraza se od ustreznih izrazov za $\mathcal{D}\mathbf{r}$ (enačba (3)) in za $\mathcal{D}^2\mathbf{r}$ (enačba (5)) razlikujeta v tem, da se v njima pojavljata relativna hitrost in pospešek namesto absolutne hitrosti in pospeška.

Z uporabo operatorja $\mathcal{D}$ lahko pospešek zapišemo v krajši obliki:

These expressions differ from the expressions for $\mathcal{D}\mathbf{r}$ (Eq. (3)) and $\mathcal{D}^2\mathbf{r}$ (Eq. (5)) in that they involve relative velocity and acceleration in place of their absolute counterparts.

Using the $\mathcal{D}$ operator we can write the acceleration in a compact form:

$$\mathbf{a} = \mathfrak{D}^2\mathbf{r} + 2\boldsymbol{\omega}\times(\mathfrak{D}\mathbf{r}) + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) + \dot{\boldsymbol{\omega}}\times\mathbf{r} \qquad (27).$$

Gibalno enačbo lahko tedaj zapišemo v obliki [5]:

Finally, the equation of motion can be put in the following form [5]

$$\rho(\mathfrak{D}^2\mathbf{r} + 2\boldsymbol{\omega}\times\mathfrak{D}\mathbf{r} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) + \dot{\boldsymbol{\omega}}\times\mathbf{r}) = \frac{\partial}{\partial s}(T\mathbf{k}) + \mathbf{f}_{\mathrm{zr}} \qquad (28).$$

To je iskana enačba gibanja preje. Prvi člen na levi pomeni relativni pospešek, podobno kakor običajni drugi odvod po času v inercialnih koordinatnih sistemih. Naslednji trije členi so sistemske (navidezne) sile, ki se pojavijo v neinercialnih vrtečih se sistemih:

1. $-\rho 2\omega\times\mathcal{D}\mathbf{r}$ pomeni Coriolisovo silo. To silo poznamo tudi na zemeljski obli: zaradi vrtenja Zemlje okoli svoje osi je tir predmetov, ki letijo vodoravno, ukrivljen proti desni na severni polobli, in proti levi na južni polobli.

This is the equation of motion of the yarn that we sought. The first left-hand term corresponds to the relative acceleration, in analogy to the more familiar second time derivative of the radius vector in inertial coordinate systems. The following three terms correspond to the system (virtual) forces that appear in non-inertial rotating systems:

1. $-\rho 2\omega\times\mathcal{D}\mathbf{r}$ corresponds to the Coriolis force. Such a force is also present on the Earth: due to the rotation of the planet, the trajectory of objects in motion in the horizontal plane deviates to the right in the northern hemisphere and to the left in the southern hemisphere.

2. $-\rho\omega\times(\omega\times\mathbf{r})$ kaže v radialni smeri navzven: to je dobro znana centrifugalna sila.
3. $-\rho\dot{\omega}\times\mathbf{r}$ ustvari sistemsko silo v sistemih, katerih kotna hitrost se spreminja s časom (na Zemlji je zanemarljiva).

Sistemske sile, ki delujejo na kratek odsek preje, so prikazane na sliki (4). Centrifugalna in Coriolisova sila sta dobro poznani, poudarili pa bi radi sistemsko silo $-\rho\dot{\omega}\times\mathbf{r}$, o kateri se do sedaj v poznani literaturi ni govorilo.

Običajni valjni navitki so sestavljeni iz plasti preje z izmenjujočimi se koti navijanja ϕ: v eni plasti je ta kot pozitiven, v naslednji pa negativen [6]. Kotna hitrost ω v ustaljenem stanju je odvisna od kota navijanja in je v preprostem približku podana z [7]:

2. $-\rho\omega\times(\omega\times\mathbf{r})$ is directed radially outward: this is the well-known centrifugal force.
3. $-\rho\dot{\omega}\times\mathbf{r}$ describes a system force in rotating frames, where the angular velocity changes with time (on Earth this force is negligible).

The system forces that act on a short segment of yarn are shown in Fig. (4). The centrifugal and Coriolis forces are well known; however, we would like to emphasize the role of the system force $-\rho\dot{\omega}\times\mathbf{r}$, which was not taken into account in the literature, to the best of our knowledge.

Typical cylindrical packages consist of layers of yarn with an alternating winding angle ϕ: in one layer this angle is positive and in the next layer it is negative [6]. The steady-state angular velocity ω depends on the winding angle, and in a simple approximation it is given by [7]:

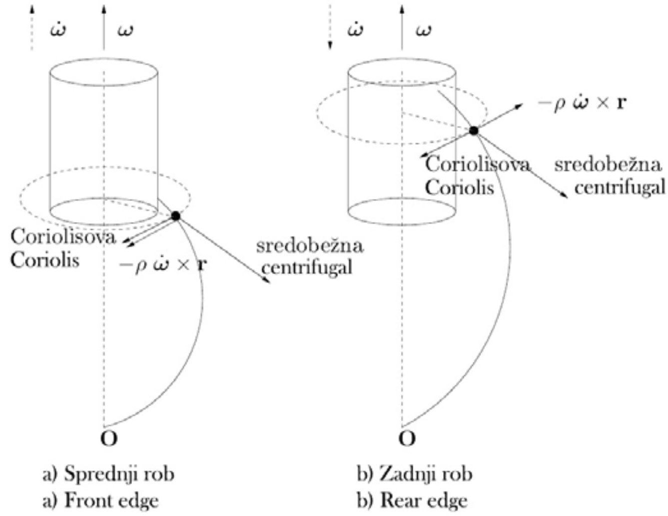
$$\omega = \frac{V}{c}\frac{\cos\phi}{1-\sin\phi} \quad (29).$$

Kotna hitrost je približno nespremenljiva v sredini navitka: večja je med odvijanjem v smeri proti zadnji strani navitka (ϕ>0) in manjša med odvijanjem v smeri proti prednji strani navitka (ϕ<0). Na obeh robovih navitka se kotna hitrost spremeni dokaj naglo: ko točka odvijanja doseže prednji rob navitka, se zveča, ko doseže zadnji rob navitka, se zmanjša. Zato sistemska sila $-\rho\dot{\omega}\times\mathbf{r}$ kaže v smeri Coriolisove sile, ko je točka dviga preje na sprednjem robu navitka, ko se kotna hitrost ω povečuje. Vektor kotnega pospeška $\dot{\omega}$ je na sliki narisan črtkano. Ko pa je točka dviga preje na zadnjem robu navitka, ta sila kaže v nasprotni smeri od Coriolisove sile, saj se tedaj kotna hitrost ω zmanjšuje in kaže vektor kotnega pospeška $\dot{\omega}$ v nasprotni smeri kakor na sprednjem robu navitka. Na sredini navitka, ko so razmere navidezno ustaljene in se s časom spreminjajo le počasi, te sile ni. (To seveda velja le za plasti z velikim številom ovojev, torej za natančno navite navitke.)

Omenjena sila ima vpliv na gibanje preje na robovih navitka, kjer se kot navijanja obrne. Tako hitra sprememba kota navijanja povzroči naglo spremembo kotne hitrosti ω, kar pomeni, da je kotni pospešek $\dot{\omega}$ velik. Zato je tudi sistemska sila $-\rho\dot{\omega}\times\mathbf{r}$ velika in spremeni dinamiko preje. Iz navidezno ustaljenih razmer pridemo tedaj v prehodno območje, ko se gibanje preje naglo spreminja. Na robovih lahko zato prihaja do nestabilnosti v obliki balona, nitka se lahko zagozdi in pretrga.

The angular velocity is approximately constant in the middle of the package: it is higher during unwinding toward the rear end of the package (ϕ>0) and lower during unwinding toward the front end of the package (ϕ<0). It changes rather abruptly at the edges of the package: it increases when the unwinding point reaches the front end of the package and it decreases when it reaches the rear end of the package. Therefore, the system force $-\rho\dot{\omega}\times\mathbf{r}$ is directed in the same direction as the Coriolis force when the lift-off point is near the front edge of the package, where the angular velocity ω is increasing. The angular acceleration vector ω is depicted in the figure by a dashed arrow. On the other hand, when the lift-off point is on the rear edge of the package, the direction of this force is opposite to the direction of the Coriolis force since the angular velocity ω is then decreasing and the angular acceleration vector $\dot{\omega}$ has the opposite direction as it has on the front edge of the package. In the middle part of the package, where the conditions are quasi-stationary and hardly change with time, this force is not present. (This is of course only true for layers with a large number of loops, i.e., for precision-wound packages.)

This force exerts an influence on the yarn motion near the edges of the package, where the winding angle changes. Such a sudden change of the winding angle leads to a rapid change of the angular velocity ω, which means that the angular acceleration $\dot{\omega}$ is high. For this reason the system force $-\rho\dot{\omega}\times\mathbf{r}$ is also substantial and it can modify the dynamics of the yarn motion. Near the edges the quasi-stationary state changes to a transient state when the yarn motion is changing rapidly. This can lead to instability of the balloon form, the yarn can jam and then break.



Sl. 4. *Sistemske sile na kratek odsek preje*
Fig. 4. *System forces acting on a segment of the yarn*

Na kratek odsek preje delujejo sredobežna, Coriolisova sila in sila $-\rho\dot{\omega}\times\mathbf{r}$.

Centrifugal, Coriolis and $-\rho\dot{\omega}\times\mathbf{r}$ forces act on a segment of the yarn.

4 POGOJ NERAZTEZNOSTI

4 CONDITION FOR INEXTENSIBILITY

Vzeli smo, da je preja neraztezna, s čimer smo imeli v mislih, da smemo zanemariti elastični raztezek preje. Poglejmo, kaj to pomeni z matematičnega vidika. Izberimo si dve bližnji točki na preji, *A* s parametrom *s* in *B* s parametrom $s+\delta s$. Ker je parameter *s* ločna dolžina od izhodišča koordinatnega sistema, ki smo ga postavili na vodilo, je dolžina preje med točkama *A* in *B* kar δs. Točki *A* in *B* povezuje vektor $\delta\mathbf{r}=\mathbf{r}_A-\mathbf{r}_B$, tako da je razdalja med točkama *A* in *B* enaka $|\delta\mathbf{r}|$. V limiti $\delta s\to 0$ je razdalja med točkama enaka dolžini preje, ki povezuje točki, zato velja:

We have assumed that the yarn is inextensible in the sense that we can neglect the elastic elongation in the yarn. We now show what this means from a mathematical point of view. We choose two nearby points on the yarn: *A* with parameter *s* and *B* with parameter $s+\delta s$. As the parameter *s* is the arc-length from the origin of the coordinate system, chosen in the guide, the length of yarn between points *A* and *B* is δs. Points *A* and *B* can be joined by a vector $\delta\mathbf{r}=\mathbf{r}_A-\mathbf{r}_B$ so that the distance between points *A* and *B* equals $|\delta\mathbf{r}|$. In the limit $\delta s\to 0$ the distance between the points is equal to the length of yarn between the points, so that:

$$|\delta\mathbf{r}| = \delta s \quad (30).$$

To lahko zapišemo tudi kot $|\partial\mathbf{r}/\partial s|=1$, s pomočjo zveze $\mathbf{x}\cdot\mathbf{x}=|\mathbf{x}||\mathbf{x}|$ pa dobimo izraz [8]:

This can also be written as $|\partial\mathbf{r}/\partial s|=1$. Using the relation $\mathbf{x}\cdot\mathbf{x}=|\mathbf{x}||\mathbf{x}|$ we obtain the expression [8]:

$$\mathbf{r}'\cdot\mathbf{r}' = 1 \quad (31).$$

Odvajamo izraz (16) po parametru *s*:

We calculate the arc-length derivative of expression (16):

$$\begin{aligned}\mathbf{r}' &= p'\mathbf{e}_p + p\mathbf{e}_p' + z'\mathbf{e}_z \\ &= p'\mathbf{e}_p + p\frac{\partial\mathbf{e}_p}{\partial\phi}\phi' + z'\mathbf{e}_z \\ &= p'\mathbf{e}_p + p\phi'\mathbf{e}_\phi + z'\mathbf{e}_z,\end{aligned} \quad (32)$$

in izračunamo skalarni produkt, iz enačbe(31) pa dobimo iskani pogoj nerazteznosti:

then we calculate the scalar product in Eq. (31). We obtain the inextensibility condition in the form:

$$\mathbf{r}' \cdot \mathbf{r}' = (p')^2 + p^2(\phi')^2 + (z')^2 = 1 \tag{33}$$

Če izračunamo odvod po ločni dolžini enačbe(21), dobimo $\phi'=\theta'$, zato smemo tudi zapisati:

By calculating the arc-length derivative of Equation (21) we obtain $\phi'=\theta'$, so we can also write

$$\mathbf{r}' \cdot \mathbf{r}' = (p')^2 + p^2(\theta')^2 + (z')^2 = 1 \tag{34}$$

5 NUMERIČNO SIMULIRANJE ODVIJANJA PREJE

5 NUMERICAL SIMULATION OF YARN UNWINDING

Tri komponente vektorske enačbe (28) in skalarna enačba (34) skupaj sestavljajo sistem štirih nelinearnih diferencialnih enačb za štiri neznane funkcije: p,θ,z, ki opisujejo obliko in gibanje preje, ter napetost T v preji. Problem bo popolnoma določen, če podamo še robne in začetne pogoje.

Pogosto je preja gosto vzporedno navita na navitkih. Tedaj se (v vrtečem se valjnem opazovalnem sistemu) razmere le malo spremenijo v času odvijanja enega navoja in smemo uporabiti navidezno ustaljeni približek. V tem primeru časovne odvode v gibalni enačbi zanemarimo, vso časovno odvisnost pa prenesemo na robne pogoje (ker se točka dviga počasi premika po navitku). Začetnih pogojev v tem primeru sploh ne potrebujemo.

Prvi robni pogoj je, da gre preja skozi vodilo v izhodišču, kar zapišemo kot $\mathbf{r}(s=0)=0$, ali $p(0)=0$, $\theta(0)=0$, $z(0)=0$. V resnici si lahko robni pogoj za θ izberemo poljubno, vendar je $\theta=0$ najbolj praktična izbira. V točki dviga mora preja biti zvezna in ne sme biti prelomljena. Od tod sledita pogoja o zveznosti:

The three components of the vector equation Eq. (28) and the scalar equation Eq. (34) constitute a system of four nonlinear differential equations for four unknown functions: p,θ,z, that describe the form and motion of the yarn and the tension T. The problem will be fully defined if the boundary and initial conditions are known.

The yarn is often densely wound in parallel on the package. In this case the conditions (as observed in the rotating cylindrical system) hardly change in the time required for unwinding one loop and the quasi-stationary approximation applies. In this case we can neglect all the time derivatives in the equation of motion and transfer the time dependence to the changing boundary conditions (because the lift-off point slowly moves on the surface of the package). Knowing the initial conditions is not necessary since we have reduced the calculation to a boundary-value problem.

The first boundary condition is given by the fact that the yarn is withdrawn through the guide, which can be mathematically expressed as $\mathbf{r}(s=0)=0$, or equivalently $p(0)=0$, $\theta(0)=0$, $z(0)=0$. In fact the boundary condition for θ can be arbitrary, but we choose $\theta(0)=0$ for convenience. In the lift-off point the curve has to be continuous with a continuous first derivative. This gives two conditions of continuity:

$$\mathbf{r}(s = s_{\mathrm{Dv}}^{+}) = \mathbf{r}(s = s_{\mathrm{Dv}}^{-}) \text{ in } \mathbf{r}'(s = s_{\mathrm{Dv}}^{+}) = \mathbf{r}'(s = s_{\mathrm{Dv}}^{+}). \tag{35}$$

Z indeksoma + oziroma - tukaj označujemo točko tik za, oziroma tik pred točko dviga. Nazadnje velja še, da je preja v točki dviga tangenta na navitek, kar zapišemo kot:

Indices + and - denote a point on the yarn just before and just after the lift-off point. Finally, we take into account that at the lift-off point the yarn is tangential to the package, which gives:

$$p'(s = s_{\mathrm{Dv}}) = 0. \tag{36}$$

Gibalne enačbe moramo integrirati numerično. Pri tem uporabljamo strelsko metodo,

The equations of motion have to be integrated numerically. For this we use the shooting method that we ki jo bomo sedaj opisali. V izhodišču s=0 si izberemo začetne približke za odvode p', θ' in z' in za napetost T (izkaže se, da te štiri količine niso med seboj neodvisne, zato zadostuje, da si izberemo le dve, na primer p' in T). Nato gibalne enačbe integriramo, dokler ne zadenemo navitka; ustavimo se lahko na primer tedaj, ko je koordinata z enaka koordinati z točke dviga. Strel je uspešen, če v končni točki v okviru vnaprej izbrane natančnosti velja $p(s_{Dv})$=c in $p'(s_{Dv})$=0, kjer je c polmer valja. Če nismo "zadeli", moramo račun ponoviti pri ustrezno popravljenih začetnih vrednostih za r' in T.

Numerični postopek smo izvedli z uporabo numeričnih rutin iz zbirke numeričnih napotkov [9]. Diferencialne enačbe smo integrirali z Runge-Kuttovo metodo, pri streljanju pa smo uporabili Powellovo metodo.

Pri iskanju optimalne oblike navitka moramo določiti ne le obliko balona, temveč moramo rešiti še problem drsenja preje po navitku, ki ga tu nismo opisali. Zaradi oprijemanja preje in zaostale napetosti preje v navitku kratek odsek preje drsi po navitku in prihaja do trenja, namesto da bi se preja vzdignila v balon takoj v točki odvijanja. V drsečem delu preje zaostala napetost pade na vrednost napetosti v balonu v točki dviga.

Tudi ta naloga se prevede na reševanje sistema diferencialnih enačb s streljanjem, rešitve pa nato zlepimo v točki dviga z uporabo zgoraj zapisanih pogojev o zveznosti. Oblika navitka določa robne pogoje v točki odvijanja (**Od** na sliki 1). Optimiranje navitkov poteka tako, da ponavljamo celoten račun za različne oblike navitkov in iščemo takšno obliko, ki vodi k najmanjši napetosti v preji. Metoda optimiranja je najbolj učinkovita za natančno navite navitke z gostim navitjem, pri katerih imajo prehodni pojavi na robovih navitka majhen učinek na celotno dinamiko odvijanja.

6 PRIMER IZRAČUNA

Na sliki 5 sta prikazana dva pogleda na "balon", ki smo ga dobili po zgoraj opisanem numeričnem postopku. Slika ustreza poenostavljenemu računu, pri katerem nismo upoštevali drsenja preje po navitku, v polni meri pa so upoštevani vplivi sredobežne in Coriolisove sile ter zračnega upora. Takšen izračun je odvisen od

now describe. At the origin s=0 we choose starting approximations for the derivatives p', θ', z' and for the tension T (it turns out that these four quantities are not mutually independent and we only need to set two quantities, for example p' and T). We integrate the differential equations until we "hit" the package; the stopping condition can, for example, be that the current coordinate z is equal to the coordinate z of the lift-off point. A shot is successful if at the final point the equations $p(s_{Dv})$=c (c is the package radius) and $p'(s_{Dv})$=0 are fulfilled within some predetermined numerical accuracy. If we "missed", we need to repeat the calculation for suitably modified starting values of p' and T.

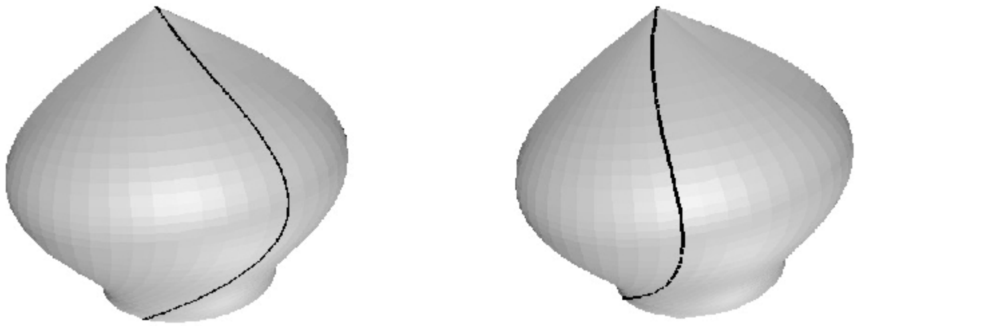
We implemented the numerical procedure using numerical routines from the Numerical Recipes library [9]. The differential equations are integrated using the Runge-Kutta method and the shooting is done using the Powell method.

For optimizing the package construction we have to determine not only the shape of the balloon, but also the sliding motion of the yarn on the surface of the package, which we have not described in this paper. Due to stiction and the residual tension of the yarn in the package, a short segment of yarn slides on the surface of the package and it rubs against it, instead of immediately lifting off in the balloon at the unwinding point. In this part of the yarn the residual tension of the yarn in the package is reduced to the value of tension in the balloon at the lift-off point.

The problem of sliding motion can also be reduced to solving a system of differential equations using the shooting method. The solutions are then glued together at the lift-off point using the conditions of the continuity that we described, while the construction of the package determines the boundary conditions at the unwinding point (**Od** on Fig. 1). The process of optimization involves repeating the calculations for different package designs and searching for the design that gives the least possible tension in the yarn. The optimisation method is most efficient for precision-wound packages with dense layers, where the transient effects at the package edges have a small effect on the overall dynamics of the unwinding process.

6 EXAMPLE OF A CALCULATION

Figure 5 represents two views of the "balloon" that we calculated using the numerical methods described above. The figure corresponds to a simplified calculation which does not take into account the sliding motion of yarn on the surface of the package. It does, however, fully take into account the effects of the centrifugal and Coriolis forces, as well as the effect of the air drag. Such



Sl. 5. *Dva pogleda na "balon": črna krivulja pomeni trenutno lego preje, siva ploskev pa je rotacijska ploskev, ki jo preja oriše v eni periodi vrtenja okoli osi. Obe sliki sta zaradi boljšega prikaza skrčeni za faktor štiri vzdolž osi vrtenja.*

Fig. 5. *Two views of the "balloon": the black curve shows the current position of the yarn, while the gray surface is the surface of revolution, that the yarn generates in one periode of its rotational motion around the axis. Both figures are scaled with a ratio of one fourth in the direction of the axis for reasons of clarity.*

enega samega brezrazsežnega parametra (p_o, brezrazsežnega koeficienta zračnega upora) in od enega robnega pogoja (navpične razdalje z_{Dv}, na kateri leži točka dviga). Parameter p_o je enak [8]:

a calculation depends on a single dimensionless parameter (p_o, the dimensionless coefficient of air friction) and on one boundary condition (the vertical distance z_{Dv} to the lift-off point). The parameter p_o is [8]

$$p_0 = 8cdc_u\rho_{zrak} / \mu_{preja} \quad (37),$$

kjer so: c polmer navitka, d premer preje, c_u koeficient zračnega upora, ρ_{zrak} gostota zraka in μ_{preja} linearna gostota preje. Izbrali smo si parameter p_o =4 in razdaljo z=12.

Dobljena rotacijska ploskev ("balon") ima trebuh, ki nastane zaradi sredobežne sile. Kakovostno podobno sliko bi dobili tudi z uporabo preprostega modela, v katerem zanemarimo Coriolisovo silo in zračni upor. Učinek teh sil pa je v resnici znaten, kar je razvidno iz oblike krivulje, ki pomeni trenutno sliko preje. Kot θ se v spodnjem delu krivulje močno spremeni in krivulja se ovija okoli balona. Račun, pri katerem Coriolisove sile ne bi upoštevali, bi dal krivuljo, ki leži v ravnini (θ =*konst*). Tako bi podcenili dolžino preje, ki ustvarja balon, velika pa bi bila tudi napaka v izračunani napetosti T.

where c is the radius of the package, d the diameter of yarn, c_u the coefficient of air friction, ρ_{air} the density of air and μ_{yarn} the linear density of the yarn. We chose p_o =4 and z=12.

The surface of revolution thus obtained (the "balloon") has a belly-shaped protrusion due to the centrifugal force. A qualitatively similar picture could be obtained using a simpler model that neglects the Coriolis force and the air drag. Nevertheless, the effect of these forces is significant, as one can see from the form of the curve that represents the snap-shot of the yarn in motion. The angle θ undergoes a rapid change in the bottom part, where the curve spirals around the balloon. A calculation which does not take into account the Coriolis force and the air drag would give a plane curve (θ=*const*). In this case the length of the yarn would be underestimated and the error in the calculated value of the tension T would also be significant.

7 SKLEP

Izpeljali in utemeljili smo sistem parcialnih diferencialnih enačb, ki opisuje gibanje preje. Enačbe veljajo za gibanje preje med poljubnim tekstilnim postopkom in so povsem splošne. V drugem delu smo se osredotočili na odvijanje preje z vzdolžnega

7 CONCLUSION

We have derived and justified a system of partial differential equations that describes the motion of the yarn. The derived equations are general and can be used to describe the yarn motion in any textile process. In the second part we focused on the over-end navitka in zapisali enačbe v vrtečem se koordinatnem sistemu, ki je bolj primeren za nadaljnjo obravnavo pri dejanskih primerih. Enačbe lahko na primer zapišemo v navidezno ustaljenem približku in jih rešimo numerično s strelsko metodo. V tem primeru se geometrijska oblika in način navitja preje na navitek kaže samo v robnih pogojih, zato je reševanje preprosto. Na ta način lahko izračunamo napetost v preji za poljubno zamišljene navitke, kar je v veliko pomoč pri iskanju navitka optimalne oblike za izbran tekstilni postopek.

unwinding of yarn from an axial package and we cast the equations in a form that is suitable for solving real problems by transforming them to a rotating coordinate system. The equations can be simplified using the quasi-stationary approximation and solved using the shooting method. In this case the geometry of the package and the type of winding appear in the boundary conditions and numerical solving is tractable. In this manner one can calculate the yarn tension for an arbitrary package design, which is helpful in optimizing the package shape for a chosen textile process.

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