Strojniški vestnik - Journal of Mechanical Engineering 51(2005)2, 90-94
UDK - UDC 681.892:534
Pregledni znanstveni članek - Review scientific paper (1.02)

Izrazi za popis upogibnega nihanja palice nespremenljivega prereza

Equations for the Flexural Vibration of a Sample with a Uniform Cross-Section

Igor Štubňa - Anton Trník

V prispevku je predstavljen kratek pregled že znanih izrazov za popis upogibnega nihanja, uporabljenih za določitev Youngovega modula in hitrosti zvoka. Predstavljen je tudi nov izraz, ki velja za vztrajnost kroženja in vpliv strižnih sil z izrazom $i_z^2[2(1+\mu)/\kappa](\partial^4 y/\partial t^2 \partial x^2)$*, v katerem je* i_z *polmer vrtenja prereza,* μ *je Poissonovo razmerje in* κ *je oblikovni faktor, ki ga je uvedel Timošenko. Krivulje porazdelitve kažejo zelo dobro ujemanje splošno uporabljanega Timošenkovega izraza in novega izraza, ki sta ga razvila Štubňa in Majerník.*

(Ključne besede: upogibno nihanje, enačbe diferencialne, izraz Timošenkov, momenti upogibni)

A short review of the known equations of flexural vibration used for determining the Young's modulus and sound velocity is presented, as well as a new equation that accounts for the rotary inertia and the influence of the shear forces with the term $i_z^2[2(1+\mu)/\kappa](\partial^4 y/\partial t^2 \partial x^2)$*, where* i_z *is the radius of gyration of the cross-section,* μ *is Poisson's ratio, and* κ *is the shape coefficient introduced by Timoshenko. The dispersion curves show a very good fit between the commonly accepted Timoshenko's equation and the new equation derived by Štubňa and Majerník.*

(Keywords: flexural vibration, partial differential equation, Timoshenko's equation, bending moments)

0 INTRODUCTION

The most convenient type of vibration used for measurement is a flexural vibration. It is easy to excite it, and the magnitude of the vibration is sufficiently high. The resonant frequency of the flexural vibration is smaller than the resonant frequency of the longitudinal or torsional vibration of a sample of the same length and cross-section. These properties of flexural vibration make it preferable for measuring the elastic modulus (or velocity of sound propagation) at elevated temperatures.

The theory of the flexural vibration of prisms and rods is based on deriving and then solving a partial differential equation of vibration for the sample. The exact solution of a three-dimensional form of the equation is extremely difficult. Fortunately, the mathematical approach to the solution of the vibration of a sample with a simple and symmetrical form can be simplified, and a reasonably exact solution can be obtained. For this reason, only the vibration of the sample with a simple uniform cross-section (circular or rectangular) serves for a measurement of the elastic parameters of solid materials.

In this paper a short review of the equations of flexural vibration commonly used for a determination of the Young's modulus or sound velocity, as well as the new equation, is presented.

1 THEORY OF FLEXURAL VIBRATION

The simplified partial equation of flexural vibration of beams with a uniform cross-section is derived on the basis of the following assumptions ([1] and [2]):

a) The amplitude of vibration is small.
b) The mass element in the direction of vibration is in equilibrium (see Fig. 1), i.e.:

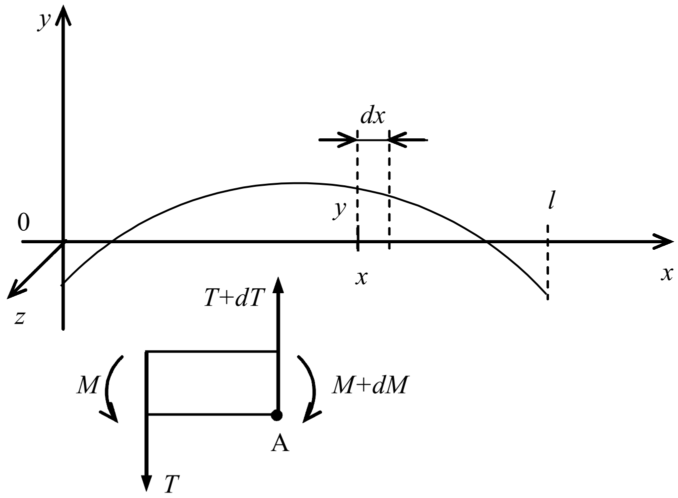

Fig. 1. *Bending line, forces and moments effecting the mass element*

$$\rho S dx \frac{\partial^2 y}{\partial t^2} = (T + dT) - T = \frac{\partial T}{\partial x} dx \qquad (1)$$

where ρ is the density of the beam material, S is the area of the cross-section, T is the shear force, t is time and x, y are coordinates.

c) The equation of the elastic line holds:

$$EJ \frac{\partial^2 y}{\partial x^2} = -M \qquad (2)$$

where M is the bending moment, E is the Young's modulus and J is the moment of inertia of the cross-section around the axis parallel with the z-axis.

d) The relationship between the shear force and the bending moment has the form:

$$\frac{\partial M}{\partial x} = T \qquad (3).$$

Eliminating the shear force T from Eq. (1) with the help of Eqs. (2) and (3) we obtain:

$$\frac{\partial^2 y}{\partial t^2} + c_0^2 i_z^2 \frac{\partial^4 y}{\partial x^4} = 0 \qquad (4)$$

where $c_0 = \sqrt{E/\rho}$ is the sound velocity (i.e., the velocity of the longitudinal wave propagation in the sample), $i_z = \sqrt{J/S}$ is the radius of gyration of the cross-section. Eq. (4) describes the vibrational motion of the sample with a sufficient exactness only when the ratio $l/d > 20$, where l is the length of the sample and d is the diameter of the cylindrical sample or thickness of the prismatic sample in the direction of vibration. The solution of Eq. (4) is the function:

$$y = y_m \exp\left[j\omega \left(t \pm \frac{x}{c} \right) \right] \qquad (5)$$

where $j = \sqrt{-1}$, $\omega = 2\pi \;\; c/\lambda$ is the angular frequency, c is the phase velocity of the flexural wave and λ is the wavelength. Substituting Eq. (5) into Eq. (4) we obtain:

$$\frac{c}{c_0} = 2\pi \left(\frac{i_z}{\lambda} \right) \qquad (6).$$

In Eq. (4) we anticipated only a displacement motion of the mass element in the direction of the y-axis. In the case of a fundamental mode vibration of a short sample (in which $l/d < 20$) the rotation of the mass element around the axis parallel with the z-axis must be taken into account. The rotation of the mass element must also be accounted for in the case $l/d > 20$ when the sample vibrates at a higher mode because the sample is divided into short parts by knots. The rotary motion of the mass element is described as (see Fig. 1):

$$\rho J dx \frac{\partial^2}{\partial t^2} \left(\frac{\partial y}{\partial x} \right) = = T dx + M - (M + dM) = T dx - \frac{\partial M}{\partial x} dx \qquad (7).$$

If we derive Eq. (7) according to x and eliminate T and M by means of Eqs. (1) and (2) we obtain an equation that includes the Rayleigh's correction (see e.g., [3]):

$$\frac{\partial^2 y}{\partial t^2} + c_0^2 i_z^2 \frac{\partial^4 y}{\partial x^4} - i_z^2 \frac{\partial^4 y}{\partial t^2 \partial x^2} = 0 \qquad (8).$$

Substituting Eq. (5) into Eq. (8) we obtain:

$$\frac{c}{c_0}=2\pi\left(\frac{i_z}{\lambda}\right)\frac{1}{\sqrt{1+4\pi^2\left(\frac{i_z}{\lambda}\right)^2}} \qquad (9).$$

As we can see from Fig. 2, the curves of the functions (6) and (9) correspond to the curve of function (15) only for a long wavelength.

Another step in the agreement between theory and experiment was made by Timoshenko [1], who proposed a correction for the effect of shear forces. Timoshenko made a hypothesis according to which the angle between the tangent to the elastic line and the x-axis is the sum:

$$\frac{\partial y}{\partial x}=\psi+\chi \qquad (10)$$

where the angles ψ and χ are connected with the shear force and the bending moment according to:

$$T=SG\kappa\chi \quad \text{and} \quad EJ\frac{\partial\psi}{\partial x}=-M \qquad (11)$$

and the moment condition of the equilibrium of the mass element is:

$$\rho Jdx\frac{\partial^2\psi}{\partial t^2}=Tdx-\frac{\partial M}{\partial x}dx \qquad (12).$$

In Eq. (11) G is the shear modulus of elasticity and κ is a constant that depends on the shape of the cross-section. From Eqs. (12), (11), (10) and (1) we obtain Timoshenko's equation [1] by the sequential elimination of the values of M, T, ψ and χ:

$$\frac{\partial^2 y}{\partial t^2}+c_0^2 i_z^2\frac{\partial^4 y}{\partial x^4}-i_z^2(1+p)\frac{\partial^4 y}{\partial x^2\partial t^2}+\frac{i_z^2}{\kappa c_s^2}\frac{\partial^4 y}{\partial t^4}=0 \qquad (13)$$

where $c_s=\sqrt{G/\rho}$ and $p=2(1+\mu)/\kappa$, and where $\mu=(E/2G)-1$ is Poisson's ratio. Timoshenko's equation describes the flexural vibration of the sample with a circular or square cross-section very well and in accordance with experimental results. For samples with a different form of cross-section Pickett proposed equation [4]:

$$\begin{aligned}&\frac{\partial^2 y}{\partial t^2}+c_0^2 i_z^2\frac{\partial^4 y}{\partial x^4}-i_z^2(1+p)\frac{\partial^4 y}{\partial x^2\partial t^2}+\\&+(i_z^2-i_y^2)\frac{\partial^4 y}{\partial x^2\partial t^2}+\frac{i_z^2}{\kappa c_s^2}\frac{\partial^4 y}{\partial t^4}=0\end{aligned} \qquad (14).$$

Eq. (14) transforms into Eq. (13) for a circular or square cross-section. However, the influence of the fourth term in Eq. (14) in the case of other cross-section shapes is very small. Substituting Eq. (5) into Eq. (13) we obtain:

$$\frac{c}{c_0}=+\sqrt{\frac{1}{2}\left(A-\sqrt{A^2-\frac{4}{p}}\right)}$$

where:

$$A=\frac{1+4\pi^2(1+p)\left(\frac{i_z}{\lambda}\right)^2}{4\pi^2 p\left(\frac{i_z}{\lambda}\right)^2} \qquad (15).$$

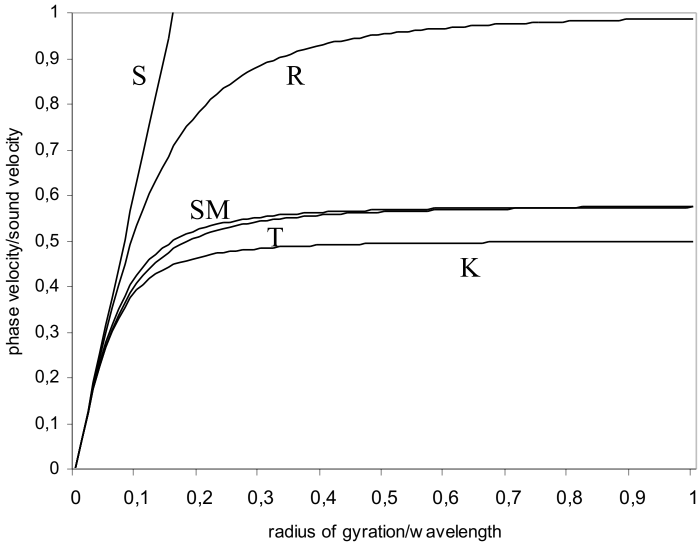

Fig. 2. *Disperse curves for the steel rod. S –for simplified equation (6), R - for equation with Rayleigh´s correction, Eq. (8), T – for Timoshenko´s equation (13), K – for Kuzmenko´s equation (16), SM – for equation derived by Štubňa and Majerník (20)*

A curve calculated from Eq. (15) is shown in Fig. 2.

From the analysis of Eq. (12) it is evident that on its left-hand side there is only an angular acceleration coming from the bending moment, and on its right-hand side there is a sum of all the moments of the forces effecting the mass element. The total angular acceleration is given by Eq. (7). Kuzmenko used Timoshenko's hypothesis (Eqs. (10) and (11)) together with Eqs. (7) and (1), [5]. Combining these equations we get:

$$\frac{\partial^2 y}{\partial t^2}+c_0^2 i_z^2\frac{\partial^4 y}{\partial x^4}-i_z^2(1+p)\frac{\partial^4 y}{\partial x^2\partial t^2}=0 \qquad (16)$$

and substituting Eq. (5) into Eq. (16) after mathematical modifications we obtain:

$$\frac{c}{c_0}=2\pi\left(\frac{i_z}{\lambda}\right)\frac{1}{\sqrt{1+4\pi^2(1+p)\left(\frac{i_z}{\lambda}\right)^2}} \qquad (17).$$

The result of Kuzmenko's attempt is shown in Fig. 2. The values c/c_0 for short wavelengths are different from those calculated by means of Eq. (15). The ratio c/c_0 for the short wavelengths must approach the value c_R/c_0, where c_R is the velocity of the propagation of Rayleigh's wave. For steel μ = 0.29 and c_R/c_0 = 0.577, [2]. To fulfil the physical requirement $c\rightarrow c_R$ when $\lambda\rightarrow 0$, it is necessary to change the coefficient from (1+*p*) in Eq. (17) to *p*. Then:

$$\lim_{\lambda\to 0}\frac{c}{c_0}=\lim_{\lambda\to 0}\frac{2\pi(i_z/\lambda)}{\sqrt{1+4\pi^2 p(i_z/\lambda)^2}}=\frac{c_R}{c_0} \qquad (18),$$

as can be seen in Fig. 2. After substituting *p* into Eq. (16) we obtain the equation:

$$\frac{c}{c_0}=2\pi\left(\frac{i_z}{\lambda}\right)\frac{1}{\sqrt{1+4\pi^2 p\left(\frac{i_z}{\lambda}\right)^2}} \qquad (19)$$

which gives a result very close to the curve of Eq. (15), see Fig. 2. We obtain the equation for phase velocity (19) from the new equation derived by Štubňa and Majerník [6]:

$$\frac{\partial^2 y}{\partial t^2}+c_0^2 i_z^2\frac{\partial^4 y}{\partial x^4}-i_z^2 p\frac{\partial^4 y}{\partial x^2\partial t^2}=0 \qquad (20)$$

which we obtain in the same way as Eq. (16) by using *p* instead of (1+*p*).

The solution for the differential equation of flexural vibration (20) can also be written in the form of a function of the type:

$$y(x,y)=Y(x)\Theta(t)=[\alpha\sinh ax+\beta\cosh ax+ +\gamma\sin bx+\delta\cos bx]\exp(j\omega t) \qquad (21)$$

where:

$$a=\frac{\omega}{c_0}\sqrt{-\frac{p}{2}+\sqrt{\frac{p^2}{4}+\left(\frac{c_0}{i_z\omega}\right)^2}}$$
$$b=\frac{\omega}{c_0}\sqrt{+\frac{p}{2}+\sqrt{\frac{p^2}{4}+\left(\frac{c_0}{i_z\omega}\right)^2}} \qquad (22)$$

The values for the bending moment and the shear force are:

$$M=-EJ\left[\frac{d^2Y}{dx^2}+Yp\frac{\omega^2}{c_0^2}\right]\exp(j\omega t)$$
$$T=-EJ\left[\frac{d^3Y}{dx^3}+\frac{dY}{dx}(1+p)\frac{\omega^2}{c_0^2}\right]\exp(j\omega t) \qquad (23)$$

which together with the solution of Eq. (21) and its derivation with respect to *x* make it possible to compile the frequency equation for given boundary conditions.

2 CONCLUSION

The simplified Eq. (4) suffices for flexural waves with a long wavelength (*i*/λ < 0.03). For this case Eqs. (8) and (16) give identical results, but they are more complicated. For flexural waves with a shorter wavelength (*i*/λ > 0.03) Eq. (13) or Eq. (20) must be used. The dispersion curves show a very good agreement between the commonly accepted Timoshenko's equation (13) and the new equation (20) derived by Štubňa and Majerník.

Acknowledgement

This work was supported by grant VEGA 1/0279/03.

3 REFERENCES

[1] Timoshenko, S. P. (1955) Vibration problems in engineering, *D. Van Nostrand Co.*, New York 1955
[2] Brepta, R., M.Prokopec (1972) Propagation of mechanical waves and shocks in solid bodies. *Academia*, Prague.
[3] Lamb, H. (1960) The dynamical theory of sound. *Gos. izd. fiz-mat literatury*, Moskva.
[4] Schreiber, E., O. Anderson, N. Soga (1973) Elastic constants and their measurement. *McGraw-Hill Book Co*., New York.
[5] Kuzmenko, V. A. (1962) *Zavodskaya Laboratoria*, 28, No. 6, 726-731.
[6] Štubňa, I., V. Majerník (1998) *Acustica – Acta Acustica*, 84, No. 6, 999-1001.

Authors' Address: Doc.Dr. Igor Štubňa
Anton Trník
Constantine the Philosopher University
Physics Department
A. Hlinku 1
SK-949 74 Nitra, Slovakia
istubna@ukf.sk

Prejeto:
Received: 12.2.2004

Sprejeto:
Accepted: 2.12.2004

Odprto za diskusijo: 1 leto
Open for discussion: 1 year