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Preskok sistema plitve osnosimetrične bimetalne lupine z uporabo nelinearne teorije

Snap-through of the System for a Shallow Axially symmetric Bimetallic Shell using Non-linear Theory

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V prispevku obravnavamo napetostne, deformacijske in stabilnostne razmere pri tankih, osnosimetričnih plitvih bimetalnih lupinah. Po teoriji drugega reda, ki upošteva ravnotežje sil na deformiranem telesu, podajamo model z matematičnim opisom geometrije sistema, premikov, napetosti in termoelastičnih deformacij. Enačbe temeljijo na teoriji velikih premikov. Kot primer predstavljamo rezultate za krogelne lupine, ki jih aproksimiramo s parabolično funkcijo. Poleg prosto položenih lupin obravnavamo tudi vrtljivo in konzolno vpete lupine ter lupine, ki so poleg segrevanja obremenjene tudi s silo v temenu. Deformacijsko krivuljo in temperaturo preskoka računamo numerično z nelinearno strelsko metodo.

(Ključne besede: lupine bimetalne, obremenitve toplotne, preskok sistema, teorija velikih premikov)

The paper deals with the stresses, strains and buckling conditions in thin, axially symmetric, shallow, bimetallic shells. Based on third-order theory, which takes into account the equilibrium state of the forces and moments that are acting on the deformed system, the paper presents a model with a mathematical description of the geometry of the system, the stresses, the thermoelastic strains and the displacements. The mathematical formulation is based on the theory of large displacements. As an example, the results for spherical, shallow shells are shown, approximated by a parabolic function. Besides simple roller-supported shells, also simple bearing-supported shells and clamped shells are discussed. The shells are loaded with temperature and/or a concentrated load acting at the top. The displacement state and the snap-through temperature are calculated numerically using a non-linear method.

(Keywords: bimetallic shell, thermal loads, stability, snap-through of the system, large displacement theory)

0 UVOD

Pospešen razvoj strojnih znanosti v zadnjih stoletjih je omogočil izdelavo različnih naprav, od razmeroma preprostih mehanizmov pa vse do zelo zapletenih strojnih naprav, ki jih človeštvo uporablja v tehnično tehnološkem postopku preoblikovanja materialnih dobrin. Čeprav so sodobne naprave po obliki, namenu in zgradbi med seboj zelo različne, pa se zaradi pomembnosti nemotenega in zanesljivega delovanja ter njihove vrednosti izraža zahteva po njihovi zaščiti pred različnimi preobremenitvami. Posebej pri strojih, ki spreminjajo eno obliko energije v drugo ter se pri tem segrevajo, je potrebno poskrbeti za zanesljivo zaščito pred toplotno

0 INTRODUCTION

The development of mechanical sciences over the centuries has enabled the production of various devices, from relatively simple devices to very complex mechanical appliances, which are used in the technical-technological process of remodelling the material goods. Although modern devices differ greatly in shape, purpose and structure, they – for the sake of their unobstructed and reliable functioning – all require protection from various forms of overloading. This is particularly so for machines that transform one type of energy into another, warming up during the process, thus requiring real protection against excessive temperature over-loading. For this preobremenitvijo. S tem namenom v naprave vgrajujemo elemente, ki opravljajo funkcijo "toplotne varovalke", tako da stroj izklopijo takoj, ko posamezen del doseže največjo še dopustno temperaturo. Zaradi zanesljivosti delovanja so se pri zaščiti pred toplotno preobremenitvijo uveljavili črtni in ploskovni bimetalni konstrukcijski elementi, katerih delovanje sloni na znanem fizikalnem dejstvu, da se telesa s povečevanjem temperature raztezajo. Idealno homogena telesa se širijo in krčijo izotropno.

V primeru bimetalnih teles, ki so izdelana iz dveh materialov z različnima temperaturnima razteznostnima koeficientoma, pa deformacije zaradi temperaturnih sprememb ne bodo več izotropne. V prispevku želimo poiskati in matematično formulirati funkcijsko zvezo med temperaturo, napetostjo in premiki bimetala. Ta zveza je med drugim odvisna tudi od geometrijskih značilk bimetala, saj se, npr. črtni bimetalni konstrukcijski elementi v primerjavi s ploskovnimi, na temperaturne spremembe različno odzivajo. Za prakso so predvsem pomembne razlike v stabilnostnih razmerah. Plitve bimetalne lupine imajo lastnost, da pri določeni temperaturi pridejo v indiferentno stanje, kar vodi v pojav, ki je v literaturi znan pod pojmom preskok sistema.

1 GEOMETRIJA SISTEMA

Na sliki 1 je predstavljena osrednja ploskev tanke bimetalne vrtilno simetrične lupine. Osnosimetrična oblika lupine nastane z vrtenjem neke funkcije okoli ordinatne osi. Obliko nedeformirane osnosimetrične lupine v Lagrangevem koordinatnem sistemu torej določa funkcija $y = y(x)$.

Zaradi spremembe temperature se lupina deformira v novo obliko, ki jo določa funkcija $Y = Y(X)$ v Eulerjevem koordinatnem sistemu. Tudi ta oblika je osnosimetrična v predpostavljenem homogenem temperaturnem polju, zaradi česar se mehanske veličine v odvisnosti od kota φ ne spreminjajo. Vektor premika $\vec{u}$ v naravnem koordinatnem sistemu (ψ,φ,r_ψ) poljubne točke P na osrednji ploskvi določa točko P'.
Torej:

purpose, elements that function as "heat cut-out" devices are installed, disconnecting the appliance as soon as a particular part reaches the maximum permissible temperature. Because of their reliability in functioning as protection from heat over-loading, linear and plane bimetallic structural elements are often used in these devices. Their working is based on the known physical fact that bodies expand with increasing temperature. Ideally, however, homogeneous objects expand and shrink in an isotropic manner.

In the case of bimetallic bodies, which are made of two materials with different linear expansion coefficients, however, the deformations due to temperature changes will no longer be isotropic. In this paper we try to find and mathematically formulate the functional connection between the temperature, the strain and the displacements of a bimetal. This connection, among other things, also depends on the bimetal's geometrical characteristics as, for instance, the linear bimetallic structural elements in comparison with the plane elements react differently to temperature changes. For practical purposes, above all, the differences in the stability conditions are important. Shallow bimetallic shells have the property that at a certain temperature they change to an indifferent state, which leads to a phenomenon known in the literature as a "snap-through of a system".

1 GEOMETRY OF THE SYSTEM

Fig. 1 shows the flexible, middle plane, of a thin, bimetallic rotationally symmetrical shell. The axially symmetric form of the shell is a result of the rotation of a function around the ordinate axis. The shape of undeformed axially symmetric shells in the Lagrange coordinate system is thus determined by the function $y = y(x)$.

Because of the change in temperature, the shell will deform into new shape, determined by the function $Y = Y(X)$ in the Euler coordinate system. This shape is axially symmetric too, providing that at each point the bimetallic shell is exposed to the same temperature change. The axially symmetric bimetallic shell within a homogeneous temperature field, however, represents an axially symmetric loading example, and because of this the physical magnitudes depending on the angle φ remain unchanged. The displacement of an optional point P on the middle plane of the shell to the point P' is determined by the displacement vector $\vec{u}$ in the natural coordinate system.
Therefore:

$$\frac{\partial}{\partial\varphi}(\;) = 0, \qquad \vec{u}\left(\psi,\varphi,r_\psi\right) = (u,v,w)$$

Ker je problem osnosimetrične narave, $v = 0$, lahko tudi pišemo:

The displacement state is axially symmetric, so we can also write:

$$\vec{u} = (u, 0, w) \tag{1}$$

Osrednja ploskev lupine je določena z enačbo $r_\psi = r_\psi(\psi)$, zaradi česar je vektor premika $\vec{u}$ funkcija kota ψ: $\vec{u} = \vec{u}(\psi)$. Opazujmo premike na tanki bimetalni osnosimetrični lupini v homogenem temperaturnem polju (sl. 1), ki je v temenu obremenjena s silo $\vec{F}_k$.

Zaradi temperaturne obremenitve se točka P, ki ima na nedeformirani lupini lego $P(x,y(x))$, premakne v lego P' s koordinatama $P'(X,Y(X))$. Premik točke P v točko P' določimo z enačbo (1), ki jo v ravnini (ψ,r_ψ) lahko tudi pišemo $\vec{u} = (u, w)$.

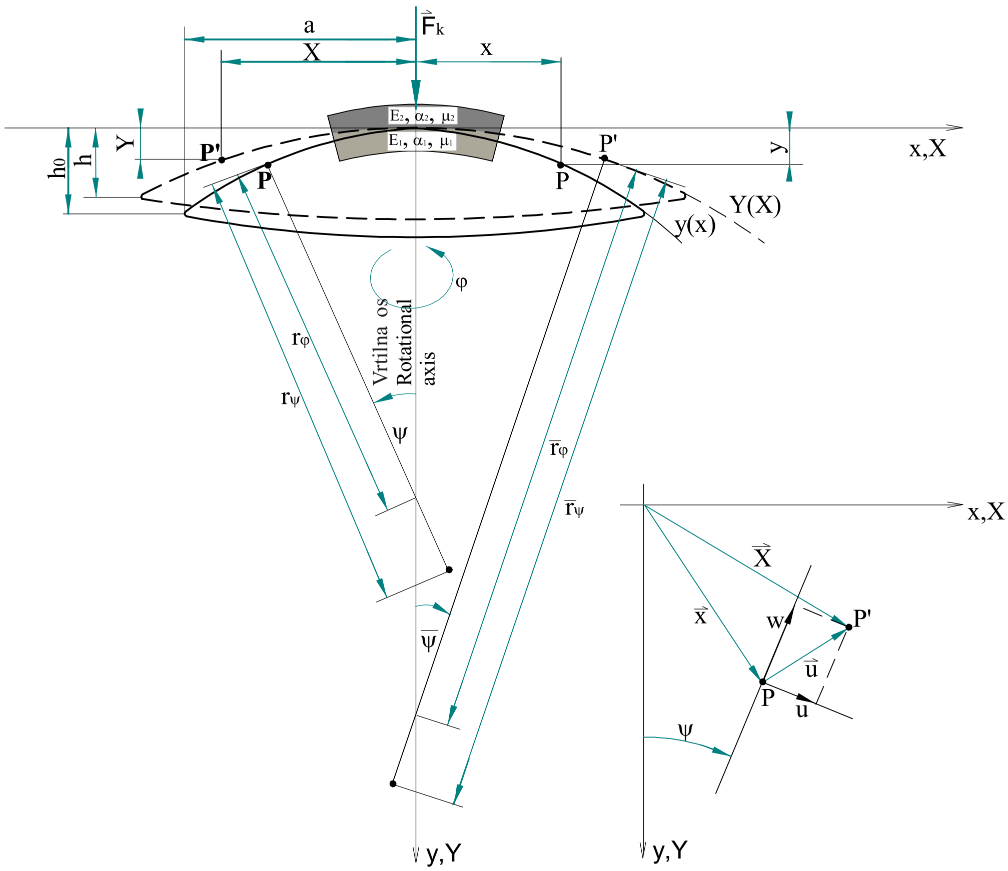
Zveza med Eulerjevim $(X,Y(X))$ in Lagrangevim $(x,y(x))$ koordinatnim sistemom je (sl. 1):

The middle plan is determined by the equation $r\psi = r\psi(\psi)$. Consequently, the displacement vector $\vec{u}$ is the function of angle ψ: $\vec{u} = \vec{u}(\psi)$. Let us observe the displacement state of a thin bimetallic axially symmetric shell in a homogeneous temperature field, Fig. 1, loaded at the top with the force $\vec{F}_k$.

Because of the temperature load, the point $P(x,y(x))$ on the undeformed shell will move to the point $P'(X,Y(X))$. The shifting of point P to the point P' is determined by Equation (1), which in the plane $(\psi,r\psi)$ can also be written as $\vec{u} = (u, w)$.

The relationship between the Euler $(X,Y(X))$ and Lagrange $(x,y(x))$ coordinates is, Fig. 1:

$$\vec{X} = \vec{x} + \vec{u}$$



Sl. 1. *Osnosimetrična lupina v homogenem temperaturnem polju in zveza med Eulerjevim in Lagrangevim koordinatnim sistemom*

Fig. 1. *The axi-symmetric shell in the homogeneous temperature field and the connection between Euler and Lagrange coordinate system*

kjer je:

where:

$$\vec{u}(x,y)=\begin{bmatrix}\cos\psi & \sin\psi\\ \sin\psi & -\cos\psi\end{bmatrix}\begin{bmatrix}u\\ w\end{bmatrix}$$

iz česar izhaja:

following that:

$$X=x+w\sin\psi+u\cos\psi \tag{2}$$

$$Y=y-w\cos\psi+u\sin\psi \tag{3}.$$

Nadalje določimo ukrivljenost nedeformirane lupine v meridialni smeri ψ in obročni smeri φ. Infinitezimalno majhen del loka na krivulji je:
V obročni smeri:

The curvature of the undeformed shell in the meridian ψ and circular φ direction is determined. An infinitesimally small part of the arc of the curve in the circular direction is:

$$ds_{\varphi}=r_{\varphi}\cdot\sin\psi\cdot d\varphi=x\cdot d\varphi$$

v meridialni smeri :

and in the meridian direction:

$$ds_{\psi}=r_{\psi}\cdot d\psi$$

Geometrija sistema na sliki 1 določa naslednja razmerja:

The geometry of the system in Fig. 1 leads to the following relations:

$$\psi=\arctan(y') \tag{4}$$

$$k_{\psi}=\frac{1}{r_{\psi}}=\frac{y''}{\sqrt{(1+y'^2)^3}} \tag{5}$$

$$k_{\varphi}=\frac{1}{r_{\varphi}}=\frac{\sin\arctan(y')}{x}=\frac{1}{x}\frac{y'}{\sqrt{1+\left(y'\right)^2}} \tag{6},$$

kjer je s k_{ψ} in k_{φ} označena ukrivljenost lupine v meridialni oziroma obročni smeri, z eno oziroma dvema črticama pa prvi oziroma drugi odvod po Lagrangevi koordinati x:

where $k\psi$ and $k\varphi$ are the curvatures of the shell in the meridian and circular directions, respectively, while the one and two apostrophes, respectively, mark the first and second derivative with respect to the independent variable x:

$$\frac{d}{dx}(\;)=(\;)',\quad \frac{d^2}{dx^2}(\;)=(\;)''$$

Zaradi deformacije lupine se kot ψ spremeni v kot $\overline{\psi}$, ukrivljenost lupine v meridialni smeri r_{ψ} oziroma obročni smeri r_{φ} smeri pa v $\overline{r}_{\psi}$ oziroma $\overline{r}_{\varphi}$. Veljajo naslednje zveze, določene z geometrijo sistema na sliki 1:

Due to the deformation of a shell the angle ψ changes into the angle $\overline{\psi}$. The curvatures of the deformed shell are $\overline{r}_{\psi}$ and $\overline{r}_{\varphi}$ in the meridian and circular directions, respectively. From the geometry of the system in Fig. 1 the following relations are determined:

$$X=\overline{r}_{\varphi}\sin\overline{\psi},\;\; \dot{Y}=\tan\overline{\psi},\;\; \overline{k}_{\psi}=\frac{1}{\overline{r}_{\psi}}=\frac{\ddot{Y}}{\sqrt{(1+\dot{Y}^2)^3}}$$

$$\overline{k}_{\varphi}=\frac{1}{\overline{r}_{\varphi}}=\frac{\sin\overline{\psi}}{X}=\frac{\sin\arctan\dot{Y}}{X}=\frac{1}{X}\frac{\dot{Y}}{\sqrt{1+\left(\dot{Y}\right)^2}}$$

Z eno piko oziroma dvema pikama nad koordinato $Y(X)$ sta označena prvi oziroma drugi odvod po Eulerjevi koordinati X:

One and two points above the coordinate $Y(X)$ mark the first and the second derivatives, respectively, with respect to the variable X:

$$\dot{Y}=\frac{dY}{dX}\;\text{ in/and }\;\ddot{Y}=\frac{d^2Y}{dX} \tag{7}.$$

Ker je koordinata Y zaradi enačbe (2) posredno funkcija Lagrangeve koordinate x, torej $Y = Y(X) = Y[X(x)]$ izračunamo oba odvoda v enačbah (7) z uporabo t.i. verižnega pravila. Po vstavitvi teh odvodov v enačbo za koordinato X in enačbi za ukrivljenost lupine v deformiranem stanju sledi:

Due to Equation (2) the coordinate Y is indirectly a function of the variable x, i.e., $Y = Y(X) = Y[X(x)]$. The derivatives in Equations (7) can be calculated by means of the so-called Chain Rule. After inserting into the Equation for the coordinate X and the Equations for the curvature of the shell in the undeformed state, it follows:

$$\bar{\psi} = \arctan\left(\frac{Y'}{X'}\right) \tag{8}$$

$$\bar{k}_{\psi} = \frac{1}{\bar{r}_{\psi}} = \frac{\begin{vmatrix} X' & Y' \\ X'' & Y'' \end{vmatrix}}{\sqrt{\left(X'^2 + Y'^2\right)^3}} \tag{9}$$

$$\bar{k}_{\varphi} = \frac{1}{X}\frac{Y'}{\sqrt{\left(X'\right)^2 + \left(Y'\right)^2}} \tag{10}$$

Diferencial dol□ine $d\bar{s}_{\psi}$ v meridialni smeri deformirane lupine je:

The differential of the length $d\bar{s}_{\psi}$ in the meridian direction of the deformed shell is:

$$d\bar{s}_{\psi} = \sqrt{dX^2 + dY^2} = dx\sqrt{X'^2 + Y'^2} = \bar{r}_{\psi} d\bar{\psi} \tag{11}$$

2 DOLOČITEV TENZORJEV DEFORMACIJ IN NAPETOSTI TER VEKTORJA PREMIKA

2 DEFINING THE STRAIN AND STRESS TENSORS AS WELL AS THE DISPLACEMENT VECTOR

Deformacijo lupine podamo z elementi vektorja premika na osrednji oziroma primerjalni ploskvi, in sicer s premikom u v meridialni ter s premikom w v prečni smeri. Osrednja ploskev se zaradi upogibnih napetosti ne deformira. Elemente deformacijskega tenzorja v krivočrtnem pravokotnem koordinatnem sistemu določimo po pravilih za spremembo tenzorjev iz enega v drugi koordinatni sistem ali pa z neposredno postavitvijo na podlagi deformiranega stanja elementarnega dela lupine, kjer upoštevamo premike na ukrivljeni ploskvi ([1] do [3]):

The shell's deformation state is shown by the displacement vector in the middle, i.e., the reference plane, and that, by the displacement of u in the meridian, and the displacement of w in the radial direction. Due to bending, the middle plane does not deform. The elements of the strain tensor in the curvilinear orthogonal coordinate system are determined by rules for tensor transformation from one into another coordinate system or by direct forming on the basis of the deformed state of the elementary part of the shell, where displacements on the curved plane are taken into account ([1] to [3]):

$$\varepsilon_r = \frac{\partial w}{\partial z} = 0 \tag{12}$$

$$\varepsilon_{\psi} = \frac{1}{r_{\psi}}\left(w + \frac{\partial u}{\partial \psi}\right) + \frac{1}{2r_{\psi}^2}\left(\frac{\partial w}{\partial \psi} - u\right)^2 = \frac{1}{r_{\psi}}\left(w + \frac{\partial u}{\partial \psi}\right) + \frac{1}{2r_{\psi}^2}\left(\frac{\partial w}{\partial \psi}\right)^2 \tag{13}$$

$$\varepsilon_{\varphi} = \frac{1}{r_{\varphi}}\left(w + \frac{\partial v}{\partial \varphi}\frac{1}{\sin\psi} + \frac{u}{\tan\psi}\right) = \frac{1}{r_{\varphi}}\left(w + \frac{u}{\tan\psi}\right) \tag{14}$$

$$\gamma_{\psi\varphi} = \frac{\partial v}{\partial \psi}\frac{1}{r_{\psi}} + \frac{\partial u}{\partial \varphi}\frac{1}{r_{\varphi}\sin\psi} - \frac{v}{r_{\psi}}\cot\psi = 0 \tag{15}$$

$$\gamma_{\psi r} = \frac{-u}{r_{\psi}} + \frac{\partial u}{\partial r} + \frac{\partial w}{\partial \psi}\frac{1}{r_{\psi}} = \frac{-u}{r_{\psi}} + \frac{\partial w}{\partial \psi}\frac{1}{r_{\psi}} \tag{16}$$

$$\gamma_{\varphi r} = \frac{-v}{r_\varphi} + \frac{\partial v}{\partial r} + \frac{\partial w}{\partial \varphi}\frac{1}{r_\varphi \cdot \sin\psi} = 0 \qquad (17).$$

V enačbah od (12) do (17) smo upoštevali predpostavko, da je lupina tanka in osnosimetrična tudi v deformiranem stanju, kakor sledi:

In Equations (12) to (17) we have assumed that the shell is also thin and axially symmetric in the deformed state, as follows:

$$\frac{\partial u}{\partial z} = \frac{\partial w}{\partial z} = 0, \quad \frac{\partial}{\partial \varphi}(\ ) = 0, \quad v = 0 \qquad (18).$$

V enačbi (13) smo za pravokotno specifično deformacijo ε_ψ v meridialni smeri upoštevali tudi nelinearni člen. Izkazalo se je namreč, da je upoštevanje nelinearnega člena in s tem teorije velikih premikov nujno za pravilnost rezultatov. Pri tem smo upoštevali, da je v nelinearnem členu komponenta u v primerjavi s komponento w zanemarljivo majhna.

Z uvedbo krajevne koordinate z, z izhodiščem na osrednji ploskvi lupine, so zaradi ukrivljenosti lupine elementi tenzorja specifičnih deformacij tudi funkcija koordinate z:

According to the third-order theory we also take into consideration the non-linear term in for the strain ε_ψ in the meridian direction. Furthermore, we take into account that the displacement u in the non-linear term is negligible compared with the displacement w.

By introducing the local coordinate z with the centre of origin in the middle plane of the shell, the elements of the strain tensor are, due to the shell's curvature, also a function of the coordinate z, as follows:

$$\varepsilon_\psi^z = \frac{\varepsilon_\psi}{1+\frac{z}{r_\psi}} + \frac{z}{1+\frac{z}{r_\psi}}\left(\frac{1}{\bar{r}_\psi} + \frac{\varepsilon_\psi}{\bar{r}_\psi} - \frac{1}{r_\psi}\right) = \varepsilon_\psi + z\left(\frac{1}{\bar{r}_\psi} - \frac{1}{r_\psi}\right) = \varepsilon_\psi + z\cdot\Upsilon_\psi \qquad (19)$$

in / and

$$\varepsilon_\varphi^z = \frac{\varepsilon_\varphi}{1+\frac{z}{r_\varphi}} + \frac{z}{1+\frac{z}{r_\varphi}}\left(\frac{1}{\bar{r}_\varphi} + \frac{\varepsilon_\varphi}{\bar{r}_\varphi} - \frac{1}{r_\varphi}\right) = \varepsilon_\varphi + z\left(\frac{1}{\bar{r}_\varphi} - \frac{1}{r_\varphi}\right) = \varepsilon_\varphi + z\cdot\Upsilon_\varphi \qquad (20),$$

kjer smo upoštevali, da je pri tankih lupinah:

where it has been taken into account that for thin shells:

$$\frac{z}{r_\psi} \cong \frac{z}{r_\varphi} \cong \frac{\varepsilon_\psi}{r_{\bar{\psi}}} \cong \frac{\varepsilon_\varphi}{r_{\bar{\varphi}}} \cong 0$$

Deformacijski in napetostni tenzor sta:

The stress and strain tensors are:

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_\psi^z & 0 & \varepsilon_{\psi r}^z \\ 0 & \varepsilon_\varphi^z & 0 \\ \varepsilon_{\psi r}^z & 0 & 0 \end{pmatrix}, \quad \sigma_{ij} = \begin{pmatrix} \sigma_\psi^z & 0 & \tau_{\psi r}^z \\ 0 & \sigma_\varphi^z & 0 \\ \tau_{\psi r}^z & 0 & 0 \end{pmatrix} \qquad (21),$$

kjer je zveza med deformacijskim in napetostnim tenzorjem [3]:

where the relationship between the stress and strain tensors [3] is:

$$\sigma_\psi^z = \frac{E}{1-\mu^2}\left(+\mu\varepsilon_\varphi^z + \varepsilon_\psi^z - (1+\mu)\alpha T\right) \qquad (22)$$

$$\sigma_\varphi^z = \frac{E}{1-\mu^2}\left(+\mu\varepsilon_\psi^z + \varepsilon_\varphi^z - (1+\mu)\alpha T\right) \qquad (23)$$

$$\tau_{\psi r}^z = G\cdot\gamma_{\psi r}^z \qquad (24)$$

Z oznako T je v enačbah (22) in (23) označena sprememba temperature glede na primerjalno temperaturo T_0, pri kateri je napetostno stanje v lupini povsod enako nič:

where T in Equations (22) and (23) denotes the change in temperature with respect to the reference temperature T_0 at which the stress state in the shell is throughout equal to zero:

$$\sigma_\psi^z(T_0,\psi,z) = \sigma_\varphi^z(T_0,\psi,z) = \tau_{\psi r}^z(T_0,\psi,z) = 0$$

3 SILE, MOMENTI IN MEHANSKO RAVNOVESJE

Slika 2 prikazuje elementarno majhen del bimetalne lupine z napetostmi, ki se pojavijo na prereznih ploskvah lupine. Zaradi napetostnega stanja, delujejo na ploskvah elementa *ABCD* sile dN_ψ, dN_φ in d$T_{\psi r}$ ter upogibna momenta dM_ψ in dM_φ:

3 EQUILIBRIUM STATE OF THE FORCES AND MOMENTS

Fig. 2 shows an elementary small part of the bimetallic shell, with stresses that appear on the cross-section of the shell's plane. The forces dN_ψ, dN_φ, dT_ψ and the bending moments dM_ψ, dM_φ on the element *ABCD* planes are:

$$dN_\psi = \int_{-h_1}^{h_2} \sigma_\psi^z \left(\frac{\overline{r}_\varphi + z}{\overline{r}_\varphi}\right) \overline{r}_\varphi \sin\overline{\psi} \cdot dz \cdot d\varphi = n_\psi \cdot \overline{r}_\varphi \sin\overline{\psi} \cdot d\varphi = n_\psi X \cdot d\varphi \tag{25}$$

$$dN_\varphi = \int_{-h_1}^{h_2} \sigma_\varphi^z \left(\frac{\overline{r}_\psi + z}{\overline{r}_\psi}\right) \overline{r}_\psi \cdot dz \cdot d\overline{\psi} = n_\varphi \overline{r}_\psi \cdot d\overline{\psi} = n_\varphi dL \tag{26}$$

$$dT_{\psi r} = \int_{-h_1}^{h_2} \tau_{\psi r}^z \left(\frac{\overline{r}_\varphi + z}{\overline{r}_\varphi}\right) \overline{r}_\varphi \sin\overline{\psi} \cdot dz \cdot d\varphi = t_{\psi r} \overline{r}_\varphi \sin\psi \cdot d\varphi = t_{\psi r} X \cdot d\varphi \tag{27}$$

$$dM_\psi = -\int_{-h_1}^{h_2} z\sigma_\psi^z \left(\frac{\overline{r}_\varphi + z}{\overline{r}_\varphi}\right) \overline{r}_\varphi \sin\overline{\psi} \cdot dz \cdot d\varphi = m_\psi \cdot \overline{r}_\varphi \sin\overline{\psi} \cdot d\varphi = m_\psi X \cdot d\varphi \tag{28}$$

$$dM_\varphi = -\int_{-h_1}^{h_2} z\sigma_\varphi^z \left(\frac{\overline{r}_\psi + z}{\overline{r}_\psi}\right) \overline{r}_\psi \cdot dz \cdot d\overline{\psi} = m_\varphi \cdot \overline{r}_\psi \cdot d\overline{\psi} = m_\varphi dL \tag{29}$$

kjer so n_ψ, n_φ in $t_{\psi r}$ enotske sile in m_ψ, m_φ enotska momenta, ki se pri tankih lupinah poenostavijo:

where n_ψ, n_φ, t_ψ denote unit forces and m_ψ, m_φ unit moments, which are simplified in the case of thin shells:

$$n_\psi = \int_{-h_1}^{h_2} \sigma_\psi^z \cdot dz \tag{30}$$

$$n_\varphi = \int_{-h_1}^{h_2} \sigma_\varphi^z \cdot dz \tag{31}$$

$$t_{\psi r} = \int_{-h_1}^{h_2} \tau_{\psi r}^z \cdot dz \tag{32}$$

$$m_\psi = -\int_{-h_1}^{h_2} z\sigma_\psi^z \cdot dz \tag{33}$$

$$m_\varphi = -\int_{-h_1}^{h_2} z\sigma_\varphi^z \cdot dz \tag{34}$$

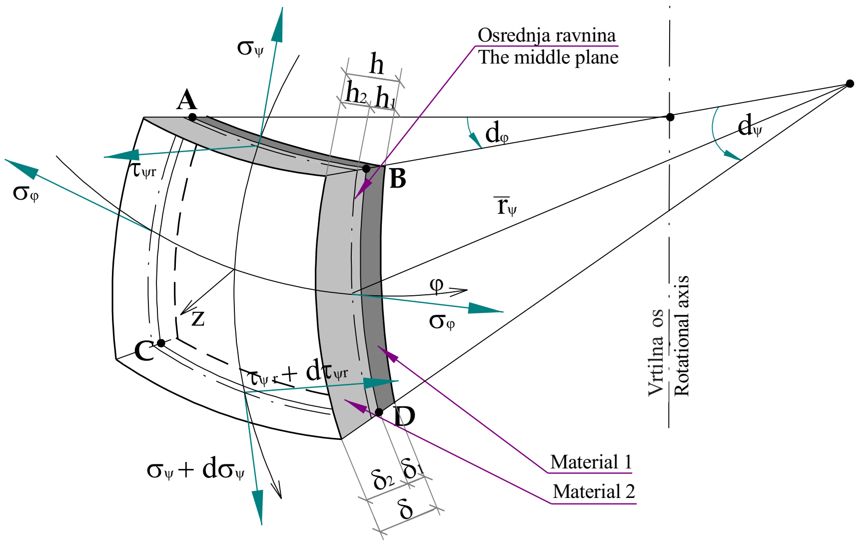
Po definiciji je osrednja ali primerjalna ploskev tista ploskev na oddaljenosti h_1 od spodnjega roba lupine, ki se zaradi upogibnih napetosti ne deformira. Lego osrednje ali primerjalne ploskve dobimo iz pogoja ravnovesja notranjih osnih sil, ki se pojavijo zaradi delovanja upogibnih napetosti:

According to the definition, the middle, i.e., the reference plane is the one that lies at the distance h_1 from the lower edge of a shell. Because of the bending stresses the middle plane does not deform. The position of the middle plane is obtained from the condition that takes into account that all inner forces, as a result of the bending moment, must be in an equilibrium state:

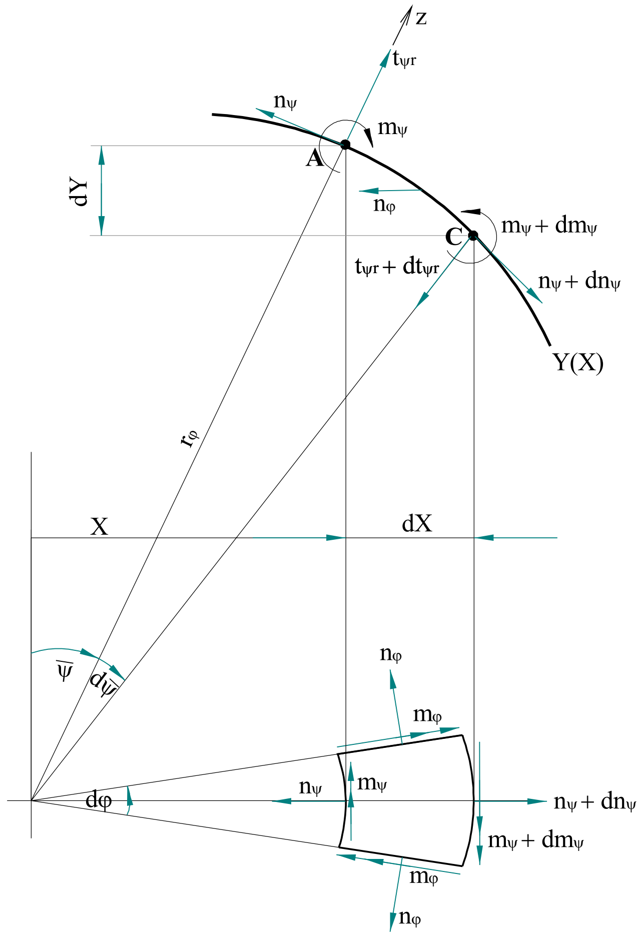
$$\int_A \sigma_\psi^z \cdot dA = \int_{-h_1}^{\delta_1 - h_1} E_1 \varepsilon_\psi^z x\,\varphi \cdot dz + \int_{\delta_1 - h_1}^{h_2} E_2 \varepsilon_\psi^z x\,\varphi \cdot dz = 0 \tag{35}$$

Specifično deformacijo ε_ψ^z računamo po enačbi (19), ki v primeru tanke lupine, ob upoštevanju, da je specifični raztezek ε_ψ osrednje ploskve enak nič, preide v obliko:

The normal strain ε_ψ^z is calculated using Equation (19), which, in the case of a thin shell and taking into account that the normal strain ε_ψ of the plane is equal to zero, takes the form:



Sl. 2. *Element na deformirani lupini*
Fig. 2. *Element on deformed shell*



Sl. 3. *Enotske sile in momenti na elementu deformirane lupine*
Fig. 3. *Unit forces and moments in the element of deformed shell*

$$\varepsilon_{\psi}^{z} = z\left(\frac{1}{\bar{r}_{\psi}} - \frac{1}{r_{\psi}}\right) \quad (36).$$

Po integraciji enačbe (35) z uporabo enačbe (36) je lega h_1 osrednje ploskve :

After integration of Equation (35) and by using Equation (36), the position h_1 of the middle plane is:

$$h_1 = \frac{E_1\delta_1^2 + 2E_2\delta_1\delta_2 + E_2\delta_2^2}{2\left(E_1\delta_1 + E_2\delta_2\right)} \quad (37).$$

Iz pogoja $h_1 + h_2 = \delta$ se določi tudi vrednost h_2. Ker je element na sliki 2 v ravnovesju, lahko zapišemo enačbe ravnovesja sil in momentov (sl. 3).

Ravnovesje sil v meridialni smeri ψ:

From the condition $h_1 + h_2 = \delta$ the value for h_2 is determined. Because the element in Fig. 2 is in an equilibrium state we can write Equations of equilibrium of the forces and moments, Fig. 3.

The equilibrium Equation of the forces in the meridian direction ψ:

$$\left(dN_{\psi} + d\left(dN_{\psi}\right)\right) - dN_{\psi}\cos d\bar{\psi} - dT_{\psi}\sin d\bar{\psi} - 2dN_{\varphi}\sin\frac{d\varphi}{2}\cdot\cos\left(\bar{\psi} + \frac{d\bar{\psi}}{2}\right) = 0 \quad (38).$$

in v prečni smeri:

and in the radial direction:

$$\left(dT_{\psi} + d(dT_{\psi r})\right) - dT_{\psi r}\cos d\psi + dN_{\psi}\sin d\bar{\psi} + 2dN_{\varphi}\sin\frac{d\varphi}{2}\cdot\sin\left(\bar{\psi} + \frac{d\bar{\psi}}{2}\right) = 0 \quad (39).$$

Ravnovesna enačba v obročni smeri je zaradi osnosimetričnega napetostno-deformacijskega stanja enaka nič. Zapišemo še upogibno momentno ravnovesje:

Due to axially symmetric deformation, the equilibrium Equation in the circular direction is equal to zero. The equilibrium Equation for the bending moments is:

$$\begin{aligned}&\left(dM_{\psi} + d\left(dM_{\psi}\right)\right) - dM_{\psi} - 2dM_{\varphi}\sin\left(\frac{d\varphi}{2}\right) - dT_{\psi r}\cos\bar{\psi}\cdot dX - \\ &dT_{\psi r}\sin\bar{\psi}\cdot dY - dn_{\psi}\sin\bar{\psi}\cdot dX + dn_{\psi}\cos\bar{\psi}\cdot dY + 2dn_{\varphi}\sin\frac{d\varphi}{2}\cdot\frac{dY}{2} = 0\end{aligned} \quad (40).$$

Upoštevajmo, da je:

Note that:

$$\sin\left(d\bar{\psi}\right) \cong d\bar{\psi}, \quad \sin\left(d\varphi\right) \cong d\varphi, \quad \cos d\bar{\psi} \cong 1$$

ter

and

$$d\left(dN_{\psi}\right) = \frac{\partial\left(dN_{\psi}\right)}{\partial\psi}d\psi = dn_{\psi}X\cdot d\varphi + n_{\psi}dX\cdot d\varphi$$

$$d\left(dT_{\psi r}\right) = \frac{\partial\left(dT_{\psi r}\right)}{\partial\psi}d\psi = dt_{\psi r}X\cdot d\varphi + t_{\psi r}dX\cdot d\varphi$$

$$d\left(dM_{\psi}\right) = \frac{\partial\left(dM_{\psi}\right)}{\partial\psi}d\psi = dm_{\psi}X\cdot d\varphi + m_{\psi}dX\cdot d\varphi$$

Tako imamo po ureditvi:

After rearranging:

$$\left(n_{\psi}X\right)' - n_{\varphi}L'\cos\bar{\psi} - t_{\psi r}X\left(\bar{\psi}\right)' = 0 \quad (41)$$

$$\left(t_{\psi r}X\right)' + n_{\varphi}L'\sin\bar{\psi} + n_{\psi}X\cdot\left(\bar{\psi}\right)' = 0 \quad (42)$$

$$\left(m_{\psi}X\right)' - m_{\varphi}L' + n_{\psi}X\left(Y'\cos\bar{\psi} - \sin\bar{\psi}\right) - t_{\psi r}X\left(Y'\sin\psi + \cos\bar{\psi}\right) = 0 \quad (43).$$

4 REŠEVANJE ENAČB ZA PRIMER PLITVIH LUPIN

4 SOLUTION OF THE EQUATIONS OF A SHALLOW SHELL

Mešani sistem enačb (2), (3), (8) do (10), (13), (14), (16), (19), (20), (22) do (24), (30) do (34) in (41)

The mixed system of Equations (2), (3), (8) to (10), (13), (14), (16), (19), (20), (22) to (24), (30) to (34) do (43) z neznankami σ_{ψ}^{z}, σ_{φ}^{z}, $\tau_{\psi r}^{z}$, ε_{ψ}^{z}, $\varepsilon_{\varphi}^{z}$, $\gamma_{\psi r}^{z}$, ε_{ψ}, ε_{φ}, $\gamma_{\psi r}$, u, w, $\bar{\psi}$, $\bar{r}_{\psi}$, $\bar{r}_{\varphi}$, n_{ψ}, n_{φ}, $t_{\psi r}$, m_{ψ}, m_{φ}, X, Y rešujemo takole:

Enačbi (19) in (20) vstavimo v enačbi (22) in (23). Enačbo (22) nato vstavimo v enačbi (30) in (33), enačbo (23) pa v enačbi (31) in (34). Po integriranju imamo:

and (41) to (43) with unknowns σ_{ψ}^{z}, σ_{φ}^{z}, $\tau_{\psi r}^{z}$, ε_{ψ}^{z}, $\varepsilon_{\varphi}^{z}$, $\gamma_{\psi r}^{z}$, ε_{ψ}, ε_{φ}, $\gamma_{\psi r}$, u, w, $\bar{\psi}$, $\bar{r}_{\psi}$, $\bar{r}_{\varphi}$, n_{ψ}, n_{φ}, $t_{\psi r}$, m_{ψ}, m_{φ}, X, Y, is solved with the following steps:

Equations (19) and (20) are inserted into Equations (22) and (23). Equation (22) is then inserted into Equations (30) and (33) and Equation (23) into (31) and (34). After integration we obtain:

$$n_{\psi} = A\varepsilon_{\psi} + \bar{A}\varepsilon_{\varphi} - B\Upsilon_{\psi} - \bar{B}\Upsilon_{\varphi} - PT \quad (44)$$

$$n_{\varphi} = \bar{A}\varepsilon_{\psi} + A\varepsilon_{\varphi} - \bar{B}\Upsilon_{\psi} - B\Upsilon_{\varphi} - PT \quad (45)$$

$$m_{\psi} = B\varepsilon_{\psi} + \bar{B}\varepsilon_{\varphi} - C\Upsilon_{\psi} - \bar{C}\Upsilon_{\varphi} - QT \quad (46)$$

$$m_{\varphi} = B\varepsilon_{\varphi} + \bar{B}\varepsilon_{\psi} - C\Upsilon_{\varphi} - \bar{C}\Upsilon_{\psi} - QT \quad (47),$$

kjer so A, $\bar{A}$, B, $\bar{B}$, C, $\bar{C}$, P in Q stalnice kakor sledi ([4] in [5]):

where A, $\bar{A}$, B, $\bar{B}$, C, $\bar{C}$, P and Q are constants as follows ([4] in [5]):

$$K1 = \frac{E_1\delta_1}{1-\mu_1^{\ 2}}, \quad K_2 = \frac{E_2\delta_2}{1-\mu_2^{\ 2}}, \quad A = K_1 + K_2, \quad \bar{A} = K_1\mu_1 + K_2\mu_2, \quad \mathrm{B} = \frac{1}{2}\left(2K_1h_1 - 2K_2h_2 - K_1\delta_1 + K_2\delta_2\right)$$

$$\bar{B} = \frac{1}{2}\left(2K_1h_1\mu_1 - 2K_2h_2\mu_2 - K_1\delta_1\mu_1 + K_2\delta_2\mu_2\right), \quad C = \frac{1}{3}\left(K_1\left(3h_1^2 - 3h_1\delta_1 + \delta_1^2\right) + K_2\left(3h_2^2 - 3h_2\delta_2 + \delta_2^2\right)\right)$$

$$\bar{C} = \frac{1}{3}\left(K_1\mu_1\left(3h_1^2 - 3h_1\delta_1 + \delta_1^2\right) + K_2\mu_2\left(3h_2^2 - 3h_2\delta_2 + \delta_2^2\right)\right), \quad P = K_1\left(1+\mu_1\right)\alpha_1 + K_2\left(1+\mu_2\right)\alpha_2$$

$$Q = \frac{1}{2}\left(K_1\left(1+\mu_1\right)\alpha_1\left(2h_1 - \delta_1\right) - K_2\left(1+\mu_2\right)\alpha_2\left(2h_2 - \delta_2\right)\right)$$

V prispevku obravnavamo plitve lupine. Pri takšnih lupinah je ukrivljenost $1/r_{\psi}$ zelo majhna, zaradi česar lahko v zgornjih enačbah poenostavimo določene izraze, saj velja:

This paper deals with shallow shells and in such shells the curvature $1/r_{\psi}$ is very small. For this reason, certain expressions in the above Equations can be simplified:

$$\psi \ll 1 \;\Rightarrow\; \psi \cong \sin\psi \cong \tan\psi = y', \quad \cos\psi \cong 1, \quad y'^2 \cong 0$$

$$\bar{\psi} \ll 1 \;\Rightarrow\; \bar{\psi} \cong \sin\bar{\psi} \cong \tan\bar{\psi} = \frac{Y'}{X'} \cong Y', \quad \cos\bar{\psi} \cong 1, \quad Y'^2 \cong 0$$

Eulerjevi koordinati (X,Y) v enačbi (2) sta s temi poenostavitvami sedaj:

Considering these simplifications, Euler's coordinates (X,Y) in Equation (2) are now:

$$X = x + wy' + u, \quad Y = y - w + uy' \quad (48).$$

Iz enačb (5), (6), (9) in (10) izhaja:

From Equations (5), (6), (9) and (10) it follows that:

$$\frac{1}{r_{\psi}} \cong y'', \quad \frac{1}{r_{\varphi}} \cong \frac{y'}{x}, \quad \frac{1}{\bar{r}_{\psi}} \cong Y'' \cong y'' - w'', \quad \frac{1}{\bar{r}_{\varphi}} \cong \frac{\dot{Y}}{X} \cong \frac{y' - w'}{x}$$

ter zato:

and thus:

$$\Upsilon_{\psi} = \frac{1}{\bar{r}_{\psi}} - \frac{1}{r_{\psi}} \cong -w'' \quad (49)$$

$$\Upsilon_{\varphi} = \frac{1}{\bar{r}_{\varphi}} - \frac{1}{r_{\varphi}} \cong \frac{-w'}{x} \quad (50)$$

in iz enačb (13) in (14):

and from Equations (13) and (14):

$$\varepsilon_{\psi} = y''w + u' + \frac{1}{2}\left(w'\right)^2 \quad (51)$$

$$\varepsilon_{\varphi} = \frac{1}{x}\left(y'w + u\right) \tag{52}$$

V enačbah (48) upoštevamo tudi, da je premik u majhen v primerjavi s premikom w, ta pa je majhen v primerjavi z Lagrangevo koordinato x, tako da sta Eulerjevi koordinati X in Y približno:

In Equations (48) we have taken into account that the displacement u is small in comparison with the displacement w, and the latter is small in comparison with the Lagrange coordinate x, so that the Euler coordinates X and Y are approximately:

$$X \cong x \tag{53}$$

$$\mathrm{Y} \cong y - w \tag{54}$$

Ravnovesne enačbe (41) do (43) so sedaj:

Thus the equilibrium Equations (41) to (43) are:

$$\left(n_{\psi}x\right)' - n_{\varphi} - t_{\psi r}xY'' = 0 \tag{55}$$

$$\left(t_{\psi r}x\right)' + n_{\varphi}Y' + n_{\psi}xY'' = 0 \tag{56}$$

$$\left(m_{\psi}x\right)' - m_{\varphi} - t_{\psi r}x = 0 \tag{57}$$

Pomnožimo sedaj enačbo (55) z Y' ter ji prištejmo enačbo (56). Tako imamo:

Let us now multiply Equation (55) with Y' and add Equation (56). Thus, we have:

$$\left(n_{\psi}x\right)' Y' + n_{\psi}xY'' + \left(t_{\psi r}x\right)' - t_{\psi r}xY'Y'' = 0$$

Zadnji člen v enačbi zanemarimo, saj je po predpostavki:

We disregard the last term in the equation, assuming that:

$$Y'Y'' = \frac{1}{2}\left(Y'^2\right)' \cong 0$$

Tako imamo po ureditvi zvezo:

In this way we obtain the relationship:

$$\left(t_{\psi r}x\right)' = -\left(n_{\psi}xY'\right)'$$

ter po integriranju:

and after integration:

$$t_{\psi r} = -n_{\psi}Y' + \frac{c}{x} \tag{58}$$

kjer je c integracijska stalnica, ki je odvisna od zunanje sile $\vec{F}_k$ v temenu lupine. Zvezo (58) vstavimo v enačbi (56) in (57):

where c is a constant of integration, which depends on the outer force $\vec{F}_k$ in the top of the shell. The connection (58) is inserted into Equations (56) and (57):

$$\left(n_{\psi}x\right)' - n_{\varphi} = 0 \tag{59}$$

$$\left(m_{\psi}x\right)' - m_{\varphi} + n_{\psi}xY' - c = 0 \tag{60}$$

Sedaj imamo sistem enajstih enačb (44) do (47), (49) do (52), (54), (59) in (60) z enajstimi neznankami $n_{\psi}, n_{\varphi}, m_{\psi}, m_{\varphi}, u, w, \varepsilon_{\psi}, \varepsilon_{\varphi}, Y, \Upsilon_{\psi}, \Upsilon_{\varphi}$. Naprej postopamo takole. Izrazimo premik u iz enačbe (52), ga odvajamo po koordinati x ter vstavimo v enačbo (51). Nastane zveza:

Now we obtained a system of 11 Equations (44) to (47), (49) to (52), (54), (59) and (60) and with eleven variables $n_{\psi}, n_{\varphi}, m_{\psi}, m_{\varphi}, u, w, \varepsilon_{\psi}, \varepsilon_{\varphi}, Y, \Upsilon_{\psi}, \Upsilon_{\varphi}$. The next steps are as follows. The displacement u is expressed from Equation (52). The derivative of the displacement u with respect to the variable x is then obtained and inserted into Equation (51). In this way we get:

$$\left(x\varepsilon_{\varphi}\right)' + \frac{1}{2}\left(w'\right)^2 = \varepsilon_{\psi} + w'y' \tag{61}$$

Iz enačb (44) in (45) izrazimo ε_ψ in ε_φ ter ju vstavimo v enačbo (61). Υ_ψ in Υ_φ nadomestimo z enačbama (49) in (59), enotsko silo n_φ pa s prvim členom v enačbi (59). S tem dobi enačba (61) obliko:

From Equations (44) and (45) are expressed ε_ψ and ε_φ, which are inserted into Equation (61). Next, Υ_ψ and Υ_φ are substituted by Equations (49) and (59), while the unit force n_φ is substituted by the first term in Equation (59). Thus, Equation takes the form:

$$\frac{1}{x}\left(x^3 n_\psi'\right)' = \frac{A^2-\bar{A}^2}{A}\left(y'w' - \frac{1}{2}(w')^2\right) + x\left(\frac{A\bar{B}-\bar{A}B}{A}\right)\left(w'' + \frac{1}{x}w'\right)' \qquad (62).$$

Sedaj v enačbo (60) vstavimo enačbi (46) in (47) ter spet izrazimo specifični deformaciji ε_ψ in ε_φ z enotskima silama n_ψ, n_φ in razlikama ukrivljenosti Υ_ψ in Υ_φ. Tako kakor prej nadomestimo Υ_ψ in Υ_φ z enačbama (49) in (50), enotsko silo n_φ pa s prvim členom v enačbi (59). Če iz enačbe (61) izrazimo še n_ψ' postane enačba (60):

Now, Equations (46) and (47) are inserted into Equation (60). The normal strains ε_ψ and ε_φ are expressed by the unit forces n_ψ, n_φ and the differences in curvature Υ_ψ and Υ_φ. Also, Υ_ψ and Υ_φ are substituted by Equations (49) and (59), and the unit force n_φ by the first term in Equation (59). If from Equation (61) n_ψ' is expressed too, Equation (60) becomes [4]:

$$x n_\psi (y-w)' = -\left(\frac{A\bar{B}-\bar{A}B}{A}\right)\left(y'w' - \frac{1}{2}(w')^2\right) - x\left(\frac{AC-B^2}{A}\right)\left(w'' + \frac{1}{x}w'\right)' - c \qquad (63).$$

Prvotni sistem, ki ga je sestavljalo 21 enačb in prav toliko neznank, smo naposled prevedli v sistem dveh diferencialnih enačb (62) in (63), ki opisujeta najbolj splošen primer, ko sta $\mu_1 \neq \mu_2$ in $\delta_1 \neq \delta_2$. Enačbi (62) in (63) poenostavimo, če imata obe plasti lupine enak Poissonov količnik $\mu_1 = \mu_2$, ker je takrat: $\bar{A} = \mu A$; $\bar{B} = \mu B$; $\bar{C} = \mu C$; $A\bar{B} - \bar{A}B = 0$. Zato sta v tem primeru enačbi (62) in (63):

The initial system, originally consisting of 21 equations and as many variables, has been finally converted into a system of two differential equations, (62) and (63), which outline the commonest example where $\mu_1 \neq \mu_2$ and $\delta_1 \neq \delta_2$. Equations (62) and (63) can be simplified if both layers of a shell have an equal Poisson's coefficient $\mu_1 = \mu_2$, because then: $\bar{A} = \mu A$; $\bar{B} = \mu B$; $\bar{C} = \mu C$; $A\bar{B} - \bar{A}B = 0$. Hence, in this case Equations (62) and (63) are:

$$\frac{1}{x}\left(x^3 n_\psi'\right)' = A\left(1-\mu^2\right)\left(y'w' - \frac{1}{2}(w')^2\right) \qquad (64)$$

in | and

$$x n_\psi (y-w)' = -x\left(\frac{AC-B^2}{A}\right)\left(w'' + \frac{1}{x}w'\right)' - c \qquad (65).$$

Z uvedbo brezrazsežne vodoravne koordinate χ:

By introducing a dimensionless horizontal coordinate χ:

$$\chi = \left(\frac{x}{a}\right)^2 \qquad (66)$$

in Wittrickovih funkcij G, F_0 in F:

and Wittrick's functions G, F_0 and F:

$$n_\psi = \frac{G(\chi)}{a^2}\left(\frac{AC-B^2}{A}\right) \qquad (67)$$

$$\frac{1}{x}y' = \frac{F_0(\chi)}{a^2}\sqrt{\frac{2\left(AC-B^2\right)}{A^2-\bar{A}^2}} \qquad (68)$$

$$\frac{1}{x}Y' = \frac{F(\chi)}{a^2}\sqrt{\frac{2\left(AC-B^2\right)}{A^2-\bar{A}^2}} \qquad (69)$$

prevedemo problem v brezrazsežno obliko [6]. Iz enačbe (66) izhajajo namreč razmerja:

the problem is converted into the dimensionless form [6]. Because of the introduction of a dimensionless coordinate χ we can write:

$$x = a\sqrt{\chi}, \quad \frac{\partial \chi}{\partial x} = \frac{2}{a}\sqrt{\chi}, \quad \left(\frac{\partial \chi}{\partial x}\right)^2 = \frac{4}{a^2}\chi, \quad \frac{\partial^2 \chi}{\partial x^2} = \frac{2}{a^2}$$

zato so posamezni členi v enačbah (64) in (65) po spremembi:

therefore, the individual terms in Equations (64) and (65), after the transformation, are:

$$\frac{1}{x}\left(x^3 n_\psi'\right)' = 3x\frac{\partial n_\psi}{\partial x} + x^2\frac{\partial^2 n_\psi}{\partial x^2} = \frac{4\chi}{a^2}\left(\frac{AC-B^2}{A}\right)\frac{d^2}{d\chi}\left(\chi \cdot G(\chi)\right)$$

$$y'w' = y'\left(y' - Y'\right) = \frac{\chi}{a^2}\left(\frac{2\left(AC-B^2\right)}{A^2\left(1-\mu^2\right)}\right)\left(F_0(\chi) - F(\chi)\right)F_0(\chi)$$

$$x\left(w'' + \frac{1}{x}(w)'\right)' = x\left(y' - Y'\right)'' + \left(y' - Y'\right)' - \frac{1}{x}\left(y' - Y'\right) = \frac{4\chi}{a^2}\sqrt{\left(\frac{2\left(AC-B^2\right)}{A^2\left(1-\mu^2\right)}\right)}\frac{d^2}{dx^2}\left(\chi \cdot \left(F_0(\chi) - F(\chi)\right)\right)$$

$$(y-w)' = Y'$$

Diferencialni enačbi (64) in (65) v brezrazsežni obliki sta tako:

Thus, the differential equations (64) and (65) in the dimensionless form are:

$$4(\chi G)'' = F_0^2 - F^2 \tag{70}$$

$$4\left(\chi\left(F - F_0\right)\right)'' = FG - \frac{c \cdot a^2}{\chi}\frac{A^2}{AC-B^2}\sqrt{\frac{1-\mu^2}{2\left(AC-B^2\right)}} \tag{71}$$

pri čemer smo z dvema črticama označili drugi odvod po koordinati χ:

where two apostrophes mark the second derivative with respect to the coordinate χ:

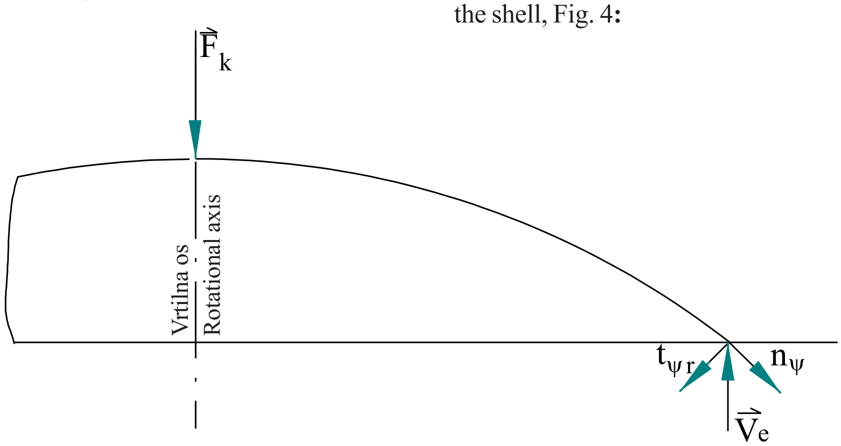
$$\frac{d}{d\chi}(\ ) = (\ )', \quad \frac{d^2}{d\chi^2}(\ ) = (\ )''$$

Odvisni brezrazsežni spremenljivki, po katerih rešujemo diferencialni enačbi (70) in (71), sta torej oblikovna funkcija trenutne oblike bimetalne lupine $F(\chi)$ ter napetostna funkcija $G(\chi)$. Integracijsko stalnico c v enačbi (71) določimo z upoštevanjem ravnovesja sil na robu lupine.

Če je lupina prosto položena, je sila podpore $\vec{V}$ na robu lupine nasprotno usmerjena in po vrednosti enaka sili $\vec{F}_k$ v temenu lupine (sl. 4):

The dependable, dimensionless variables with which we are solving the differential equations (70) and (71) are thus the formative function $F(\chi)$ of the present form of the bimetallic shell and the stress function $G(\chi)$. The constant of integration c in Equation (71) is determined by taking into account the equilibrium of forces at the edge of a shell.

If the shell is simply roller supported, the force $\vec{V}$ at the edge of a shell is pointing in the opposite direction and is equal to the force $\vec{F}_k$ in the top of the shell, Fig. 4**:**



Sl. 4. *Obremenitev prosto polo□ene lupine s silo* $\vec{F}_k$
Fig. 4. *Loading of simply-roller supported shell with the force* $\vec{F}_k$

$$\vec{F}_k + \vec{V} = \vec{F}_k + (2\pi a)\vec{V}_e = 0 \quad (72),$$

kjer je z $\vec{V}_e$ označena enotska navpična sila na robu lupine. Enačba (72) je v komponentni obliki:

where $\vec{V}_e$ denotes the vertical unit force at the edge of the shell. Equation (72) in component form is:

$$F_k + \left(t_{\psi r} + n_\psi \cdot \left.\frac{dY}{dx}\right|_{x=a}\right)2\pi a = 0, \quad -t_{\psi r} \cdot \left.\frac{dY}{dx}\right|_{x=a} + n_\psi = 0 \quad (73).$$

Rešitev sistema enačb (73) je potem, ko upoštevamo tudi plitvost:

When also taking into account the shallowness, the solution of the system of Equations (73) is:

$$n_\psi(a) = \frac{-F_k}{2\pi a} \cdot \left.\frac{dY}{dx}\right|_{x=a} \quad (74)$$

$$t_{\psi r}(a) = \frac{-F_k}{2\pi a} \quad (75).$$

Vstavimo enačbi (74) in (75) v enačbo (58) ter izrazimo stalnico c:

Equations (74) and (75) are inserted into Equation (58) and the constant c is expressed by:

$$c = \frac{-F_k}{2\pi} \quad (76).$$

Če na lupino ne deluje zunanja sila $\vec{F}_k$ je integracijska stalnica c v enačbi (76) enaka nič, diferencialni enačbi (70) in (71) pa se poenostavita v obliko, ki jo je v [6] zapisal W. H. Wittrick:

If the external force $\vec{F}_k$ does not act upon the shell, the integrating constant c in Equation (76) is equal to zero, whilst the differential Equations (70) and (71) are simplified into a form, according to W. H. Wittrick in [6]:

$$4(\chi G)'' = F_0^2 - F^2 \quad (77)$$

$$4\left(\chi\left(F - F_0\right)\right)'' = FG \quad (78).$$

5 ANALIZA RAZMER PRI KROGELNIH LUPINAH

5 ANALYSIS OF THE CIRCUMSTANCES IN SPHERICAL SHELLS

V enačbah (70) in (71) se pojavlja oblikovna funkcija začetne oblike lupine $F_0(\chi)$. Ta je odvisna od funkcije $y = y(x)$, ki opisuje osrednjo ploskev začetne, nedeformirane oblike lupine. V primeru krogelnih lupin, katerih teme je postavljeno v izhodišče koordinatnega sistema, je ta funkcija v posredni obliki: $x^2 + y^2 = 2yR$. Zaradi plitvosti lupine zanemarimo drugi člen ter dobimo po zamenjavi spremenljivke x v brezrazsežno spremenljivko χ:

In Equations (70) and (71) occurs the formative function $F_0(\chi)$ of the initial shape of the shell. It depends on the function $y = y(x)$, which describes the middle plane of the initial, undeformed shape of the shell. However, in the case of spherical shells, whose crowns are set at the beginning of the coordinate system, this function is in the implicit form $x^2 + y^2 = 2yR$. Because of the shallowness of the shell, the second term is neglected, so that after the substitution of the variable x into the dimensionless variable χ, we obtain:

$$y = \frac{x^2}{2R} = \frac{a^2\chi}{2R} = h_0\chi \quad (79).$$

Oblikovna funkcija začetne oblike lupine je po enačbi (68):

According to Equation (68) the formative function of the initial shape of the shell is:

$$F_0 = 2h_0 A\sqrt{\frac{1-\mu^2}{2\left(AC - B^2\right)}} = \text{konst} \quad (80).$$

Če upoštevamo, da je začetna oblikovna funkcija F_0 nespremenljiva, sta diferencialni enačbi

When taking into account that the initial formative function F_0 is a constant, the differential

(70) in (71):

Equations (70) and (71) are:

$$4G''\chi + 8G' = F_0^2 - F^2 \quad (81)$$

$$4F''\chi + 8F' = FG - \frac{c \cdot a^2}{\chi} \frac{A^2}{AC - B^2} \sqrt{\frac{1-\mu^2}{2\left(AC - B^2\right)}} \quad (82).$$

Diferencialni enačbi (81) in (82) najlažje rešimo za primer, ko lupina ni obremenjena z zunanjo silo $\vec{F}_k$, ker je takrat v enačbi (82) stalnica *c* enaka nič. Če je lupina prosto položena, so napetosti in momenti na njenem robu enaki nič, v temenu lupine pa imajo napetosti in momenti le končne vrednosti. Za napetostno funkcijo $G(\chi)$ veljata torej robna pogoja:

The easiest way to solve Equations (81) and (82) is in the case where the shell is not loaded by an external force $\vec{F}_k$ because then the constant *c* in Equation (82) is equal to zero. If the shell is simply roller supported, the stresses and moments at its edges are equal to zero, while the stresses and moments in the top of the shell have only limited values. For the stress function $G(\chi)$ the following boundary conditions thus hold:

$$G(1) = 0, \quad G(0) \neq \infty \quad (83).$$

Robna pogoja za oblikovno funkcijo $F(c)$ dobimo iz enačbe (46), ki jo zapišemo v brezrazsežni obliki. Iz enačbe (44) in (45) izrazimo ε_ψ in ε_φ ter vrednosti vstavimo v enačbo (46) za enotski moment m_ψ. Spremembo ukrivljenosti Υ_ψ in Υ_φ nadomestimo z enačbama (49) in (50) prvi in drugi odvod premika *w* pa izrazimo z odvodom enačbe (54). Če nadalje spet vzamemo enak Poissonov koeficient za obe plasti bimetalne lupine se enotski moment m_ψ izraža:

The boundary conditions for the formative function $F(\chi)$ are obtained from Equation (46), written in dimensionless form. From Equation (44) we express ε_ψ and insert the value into Equation (46) for the unit moment m_ψ. The curvature differences Υ_ψ and Υ_φ are then replaced by Equations (49) and (50), while the first and second derivatives of the displacement *w* are expressed by the derivative of Equation (54). In the case of an equal Poisson's ratio for both layers of the bimetallic shell, the unit moment m_ψ runs as follows:

$$m_\psi = \frac{C}{a^2}\sqrt{\frac{2C}{A(1-\mu^2)}} \cdot \Big(F_0(1+\mu) - F(\chi)(1+\mu) - 2\chi F'(\chi)\Big) - QT \quad (84),$$

kadar velja tudi:

in the special case where:

$$\delta_1 = \delta_2 = \frac{\delta}{2} \text{ in/and } E_1 = E_2 = E \quad (85).$$

Ker je na robu lupine poleg enotske sile n_ψ ničen tudi enotski moment m_ψ, izrazimo iz (84) povezavo med temperaturo lupine *T* in trenutno oblikovno funkcijo *F*:

At the edge of the shell, besides the unit force n_ψ, also the unit moment m_ψ is equal to zero. So, the relationship between the shell's temperature *T* and the present formative function *F* can be derived from Equation (84):

$$T = T_m\left(1 - \frac{1}{F_0}\left(F(1) + \frac{2}{1+\mu}F'(1)\right)\right) = T_m \cdot \tau \quad (86),$$

kjer je *t* brezrazsežna funkcija temperature, T_m pa nespremenljiva:

where τ is the dimensionless function of the temperature and T_m is a constant:

$$T_m = \frac{2h_0(1+\mu)\left(AC - B^2\right)}{a^2(AQ - BP)} \quad (87).$$

Za oblikovno funkcijo *F* sta torej robna pogoja:

Thus, the boundary conditions for the formative function *F* are:

$$\tau = 1 - \frac{1}{F_0}\left(F(1) + \frac{2}{1+\mu}F'(1)\right), \quad F(0) \neq \infty \quad (88).$$

Kadar je bimetalna lupina takšna, da veljajo enačbe (85), potem je iz enačbe (37): $h_1 = h_2 = \delta_1 = \delta_2 = \delta/2$. Če slednje upoštevamo pri izračunu stalnic $A, \bar{A}, B, \bar{B}, C, \bar{C}, P$ in Q, se enačbi (80) in (87) za nespremenljivi F_0 in T_m poenostavita:

Let us take into consideration the case where Equations (85) are fulfilled. Consequently, from Equation (37) it follows that $h_1 = h_2 = \delta_1 = \delta_2 = \delta/2$. If what was stated above is considered in the calculation of the constants $A, \bar{A}, B, \bar{B}, C, \bar{C}, P$ and Q, Equations (80) and (87) for the constants F_0 and T_m are simplified:

$$F_0 = \frac{2\sqrt{6}\cdot h_0}{\delta}\sqrt{1-\mu^2}, \quad T_m = \frac{2\delta}{3R(\alpha_1-\alpha_2)} = \frac{2\delta^2 F_0}{3\sqrt{6}\cdot a^2\sqrt{1-\mu^2}\cdot(\alpha_1-\alpha_2)} \quad (89).$$

Za numerično reševanje problema pa robna pogoja $G(0) \neq \infty$ in $F(0) \neq \infty$ nista primerna, zato uvedemo novi spremenljivki $g(\chi)$ in $f(\chi)$:

For a numerical solution of the problem, however, the boundary conditions $G(0) \neq \infty$ and $F(0) \neq \infty$ are not suitable; hence new variables $g(\chi)$ and $f(\chi)$ have been introduced:

$$f = \chi \cdot F, \quad g = \chi \cdot G \quad (90).$$

Po spremembi diferencialnih enačb (81), (82) in robnih pogojev (83) in (88) ter upoštevanju, da sta funkciji G in F v $\chi = 0$ omejeni, imamo naslednji problem robnih vrednosti:

After transforming the differential equations (81) and (82), the boundary conditions (83) and (88), and taking into consideration that functions G and F in $\chi = 0$ are limited, we obtain the following boundary-value problem:

$$\begin{gathered} 4g'' = F_0^2 - \frac{f^2}{\chi^2}, \quad 4f'' = \frac{f\cdot g}{\chi^2} \\ g(0) = g(1) = f(0) = 0, \quad \tau = 1 - \frac{1}{F_0(1+\mu)}\left(2f'(1) - f(1)\cdot(1-\mu)\right) \end{gathered} \quad (91).$$

Prevedemo ga v sistem navadnih diferencialnih enačb prvega reda:

It is then converted into a system of ordinary differential equations of the first order:

$$\begin{gathered} y_1' = y_2, \quad y_2' = \frac{1}{4}\left(F_0^2 - \frac{y_3^2}{\chi^2}\right), \quad y_3' = y_4, \quad y_4' = \frac{1}{4}\frac{y_1 y_3}{\chi^2} \\ y_1(0) = y_1(1) = y_3(0) = 0, \quad \tau = 1 - \frac{1}{F_0(1+\mu)}\left(2y_4(1) - y_3(1)\cdot(1-\mu)\right) \end{gathered} \quad (92),$$

če uvedemo zamenjavo:

when the substitution is introduced:

$$g = y_1, \quad g' = \frac{dg}{d\chi} = y_2, \quad f = y_3, \quad f' = \frac{df}{d\chi} = y_4$$

Sistem enačb (92) s temperaturo τ v območju $0 \le \tau \le 2$ smo rešili z uporabo nelinearne strelske metode. Izbrali smo približni vrednosti $y_2(1)$ in $y_4(1)$ ter izračunali približne vrednosti funkcij g in f po običajni enokoračni metodi Runge Kutta 4. reda. Določanje natančnejših vrednosti $y_2(1)$ in $y_4(1)$ je potekalo po Newtonovi metodi reševanja nelinearnih enačb. Ker je sistem enačb (92) v točki $\chi = 0$ singularen, smo odstopanje približnih vrednosti g in f od danih robnih pogojev v $\chi = 0$ računali v točki $\chi = \chi_0 = 10^{-10}$. Deformacijo lupine smo zapisali z razmerjem ξ med trenutno višino deformirane lupine in začetno višino nedeformirane lupine:

The system of Equations (92) with a temperature τ in the interval $0 \le \tau \le 2$ was solved using the non-linear shooting method. We took approximate values for $y_2(1)$ and $y_4(1)$ and calculated rough values for the functions g and f using the classical one-step Runge Kutta method of the fourth order. For defining more exact values of $y_2(1)$ and $y_4(1)$, the Newton method of solving non-linear equations was used. And since the system of Equations (92) at the point $\chi = 0$ is singular, the digressions from the approximate values g and f from the given boundary conditions in $\chi = 0$ were calculated at the point $\chi = \chi_0 = 10^{-10}$. The shell's deformation was recorded with the ratio ξ between the present height of the deformed shell and the initial height of the undeformed shell:

$$\xi = \frac{h}{h_0} = \frac{Y(1)}{y(1)} = \frac{1}{F_0}\int_{\chi_0}^{1}\frac{1}{\chi}f(\chi)\cdot d\chi \quad (93).$$

Integral v (93) smo računali numerično. Zaradi singularnosti v $\chi = 0$ je integracija potekala od $\chi = 1$ do $\chi = \chi_0$. Tako smo izračunali razmerje višin ξ pri različnih temperaturah τ. Problem robnih vrednosti (91) lahko rešimo tudi po metodi, ki jo je v [6] predlagal W. H. Wittrick. Ker sta funkciji g in f v točki $\chi = 0$ singularni, ju v okolici te točke zapišemo v obliki potenčne vrste, tako da zadostimo robnim pogojem v $\chi = 0$:

The integral (93) was solved numerically. Due to the singularity in $\chi = 0$ the integration occurred from $\chi = 1$ to $\chi = \chi_0$. Consequently, we have calculated the values of ratio ξ at various temperatures τ. The boundary value problem (91) could be solved too using the method proposed by W. H. Wittrick in [6]. Because the functions g and f at the point $\chi = 0$ are singular, they have been written in the form of power series about the point $\chi = 0$ in such a way as to meet the boundary conditions in $\chi = 0$:

$$g = \sum_{n=0}^{\infty} g_n \chi^{n+1}, \quad f = \sum_{n=0}^{\infty} f_n \chi^{n+1} \qquad (94).$$

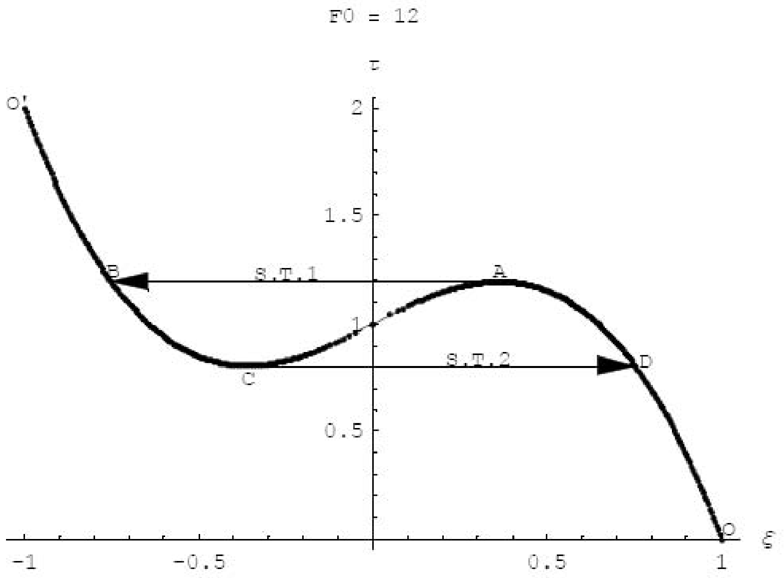
Po vstavitvi potenčnih vrst (94) v diferencialni enačbi robnega problema (91) ter primerjavi koeficientov pri enakih potencah spremenljivke χ na obeh straneh enačb, dobimo razmerja med koeficienti:

After inserting the power series (94) into the boundary-value problem (91) and comparing the coefficients in equal exponents of the variable χ on both sides of the equations, we obtain the relations between the coefficients:

$$\begin{aligned} g_1 &= \frac{F_0^2 - f_0^2}{8}, \quad f_1 = \frac{f_0 g_0}{8} \\ g_2 &= \frac{-2 f_0 f_1}{24}, \quad f_2 = \frac{f_0 g_1 + f_1 g_0}{24} \\ g_3 &= \frac{-2 f_0 f_2 - f_1^2}{48}, \quad f_3 = \frac{f_0 g_2 + f_1 g_1 + f_2 g_0}{48} \\ g_4 &= \frac{-2 f_0 f_3 - 2 f_1 f_2}{80}, \quad f_4 = \frac{f_0 g_3 + f_1 g_2 + f_2 g_1 + f_3 g_0}{80} \quad \text{itn./etc.} \end{aligned} \qquad (95).$$

Sistem enačb (95) je rekurziven. Pri izbranem g_0 in f_0 so določeni vsi nadaljnji koeficienti g_n in f_n funkcije g in f. Koeficienta g_0 in f_0 seveda izberemo tako, da funkciji g in f na robu lupine v $\chi = 1$ zadoščata robnima pogojema v enačbi (91). Za izbrani koeficient g_0 smo izbrali neko začetno vrednost koeficienta f_0 ter v območju $0 \le \chi \le 0{,}05$ določili člene potenčne vrste funkcij g in f. S potenčnima vrstama smo izračunali funkcijske vrednosti $g(0{,}05)$, $g'(0{,}05)$, $f(0{,}05)$ in $f'(0{,}05)$ ter s temi začetnimi vrednostmi numerično izračunali vrednosti funkcij g in f v območju $0{,}05 < \chi \le 1$. Ker je izbrana začetna vrednost koeficienta f_0 le približek, funkcija g v točki $\chi = 1$ odstopa od robnega pogoja v enačbi (91). Po Newtonovi metodi smo zato določili novo, natančnejšo vrednost koeficienta f_0, ter postopek ponovili tolikokrat, da je postala absolutna vrednost funkcije $g(1)$ natančna do vnaprej predpisane vrednosti. Prednost tega numeričnega postopka pred prej opisanim je v tem, da z Newtonovo metodo ali bisekcijo računamo vrednost samo ene neznanke, in sicer vrednost koeficienta f_0 oziroma $y_3(0{,}05)$, medtem ko je treba pri strelski metodi izračunati vrednosti dveh

The system of Equations (95) is recursive. By selecting g_0 and f_0 all further coefficients g_n and f_n of the function g and f have been determined. Clearly, the coefficients g_0 and f_0 are selected in such a way that the functions g and f at the edge of a shell in $\chi = 1$ meet the boundary conditions in Equation (91). For the selected coefficient g_0 we have taken an initial value for the coefficient f_0 and in the interval $0 \le \chi \le 0{,}05$ determined the elements of the power series' for functions g and f. Then we have calculated, using power series, the functional values $g(0{,}05)$, $g'(0{,}05)$, $f(0{,}05)$ and $f'(0{,}05)$. These initial values were used to compute the numerical values of the functions g and f in the interval $0{,}05 < \chi \le 1$. Because the selected initial value for the coefficient f_0 is only an approximation, the function g at the point $\chi = 1$ deviates from the boundary condition in Equation (91). Using Newton's method we have then determined a new, more exact value for the coefficient f_0 and kept repeating the procedure until the absolute value of the function $g(1)$ was as precise as set up initially. The advantage of this numerical procedure over the one described before is that by using Newton's method or bisection we calculate the value of only one variable $y_3(0{,}05)$, while with the shooting method it is necessary to calculate the values of two variables, $y_2(1)$



Sl. 5. *Funkcija $\tau = \tau(\xi)$ za primer osnosimetrične lupine s $F_0 = 12$ in $\mu = 1/3$, ki izkazuje pojav preskoka sistema med točkama AB ob segrevanju in točkama CD ob ohlajanju*

Fig. 5. *The function $\tau = \tau(\xi)$ as an example of axi-symmetric shell for $F_0 = 12$ and $\mu = 1/3$ expressing the phenomenon of a snap-through system between points AB in the process of heating up and the points CD in the process of cooling*

spremenljivk $y_2(1)$ in $y_4(1)$. Po obeh numeričnih postopkih smo dobili enake rezultate.

Slika 5 prikazuje razmere v lupini s Poissonovim koeficientom $\mu = 1/3$ in začetno oblikovno funkcijo $F_0 = 12$. Graf funkcije brezrazsežne temperature t v odvisnosti od razmerja višin ξ predstavlja stabilnostne razmere ob temperaturnem obremenjevanju bimetalne lupine .

V začetnem, temperaturno neobremenjenem stanju $\tau = 0$, v točki $O(1,0)$ je razmerje višin ξ enako ena. S povečevanjem brezrazsežne temperature t se to razmerje zmanjšuje. Kakor je razvidno s slike 5, je območje na krivulji med točko O in točko $A(\xi_{p1}, \tau_{p1}) = A(0{,}360;1{,}195)$, kjer ima funkcija $\tau(\xi)$ lokalni vrh, območje stabilnega ravnovesja. Do preskoka lupine bo torej prišlo v točki A pri temperaturi $\tau_{p1} = 1{,}195$, ker je korak med točko A in točko $C(\xi_{p2}, \tau_{p2}) = C(-0{,}360;0{,}805)$, kjer ima funkcija lokalni dol, območje nestabilnega ravnovesja. Po preskoku bo lupina zavzela novo ravnovesno lego v točki $B(-0{,}752;1{,}195)$ pri temperaturi $\tau = 1{,}195$. Pri nadaljnjem segrevanju lupine razmerje ξ še naprej zmanjšuje.

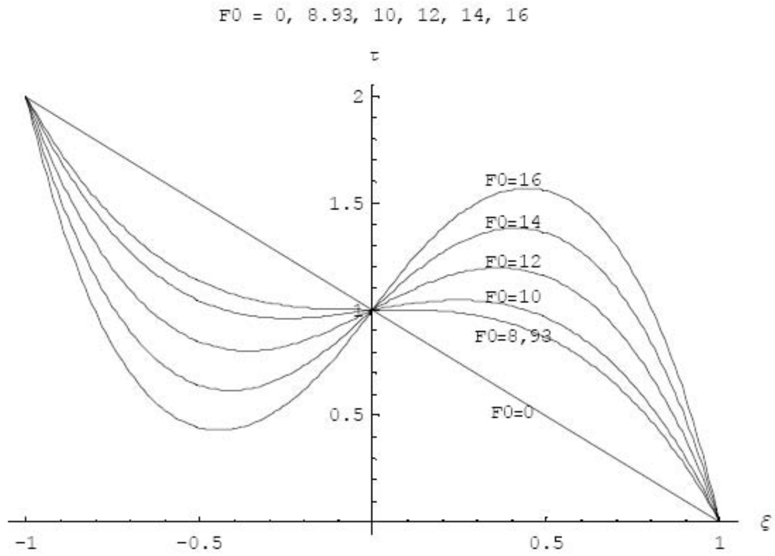
Pri ohlajanju lupine imamo nasproten pojav in v točki C pri temperaturi $\tau_{p2} = 0{,}805$, ponoven preskok. Tokrat lupina preskoči v ravnovesno lego v točki $D(0{,}752;0{,}805)$ pri temperaturi $\tau = 0{,}805$. S

and $y_4(1)$. Nevertheless, with both of these numerical methods we obtained the same results.

Fig. 5 shows the condition for the shell with Poisson's ratio $\mu = 1/3$ and the initial formative function $F_0 = 12$. The graph of the function of dimensionless temperature τ depending on the ratio of heights ξ represents the stability circumstances during the shell's temperature load.

In the initial state with temperature $\tau = 0$, at point $O(1,0)$ the ratio of heights ξ is equal to one. By increasing the dimensionless temperature τ this ratio is decreasing. As shown in Fig. 5, the portion of the curvature between point O and point $A(\xi_{p1}, \tau_{p1}) = A(0.360,1.195)$ where the function $\tau(\xi)$ has the local maximum is the range of stable equilibrium. Hence, the snap-through of the shell will happen at point A at temperature $\tau_{p1} = 1.195$ because the interval between point A and point $C(\xi_{p2}, \tau_{p2}) = C(-0.360,0.805)$where the function has the local minimum is the range of unstable equilibrium. After the snap-through, the shell will take a new equilibrium state at point $B(-0.752,1.195)$ at temperature $\tau = 1.195$. In the course of the shell's further heating up, the ratio of heights ξ continues to decrease.

However, in the cooling down of the shell the reverse situation occurs and at point C at temperature $\tau_{p2} = 0.805$, another snap-through happens. This time the shell snaps into the equilibrium state at point



Sl. 6. *Stabilnostne razmere pri lupinah z različnimi vrednostmi funkcije F_0 in μ = 1/3*
Fig. 6. *Snap-through behaviour for shells with various values of function F_0 and μ = 1/3*

Preglednica 1. *Temperatura in lega preskoka lupine za različne vrednosti funkcije F_0 in μ = 1/3*
Table 1. *Temperature and position of the shell's snap-through for different values of function F_0 and μ = 1/3*

F_0	$8,93$	10	12	14	16
$S.T.1$	$\tau_{p1}=1$ $\xi_{p1}=0$	$\tau_{p1}=1,043$ $\xi_{p1}=0,248$	$\tau_{p1}=1,195$ $\xi_{p1}=0,361$	$\tau_{p1}=1,381$ $\xi_{p1}=0,411$	$\tau_{p1}=1,567$ $\xi_{p1}=0,445$
$S.T.2$	$\tau_{p2}=1$ $\xi_{p2}=0$	$\tau_{p2}=0.957$ $\xi_{p2}=-0,248$	$\tau_{p2}=0,805$ $\xi_{p2}=-0,361$	$\tau_{p2}=0,619$ $\xi_{p2}=-0,411$	$\tau_{p2}=0,433$ $\xi_{p2}=-0,445$

ponovnim segrevanjem lupine do temperature prvega preskoka τ_{p1} = 1,195 lahko celoten krog preskokov lupine ponovimo.

Stabilnostne razmere pri lupinah z drugačnimi vrednostmi začetne oblikovne funkcije F_0 so prikazane na sliki 6, pregledniče vrednosti za temperaturo preskoka τ_p in razmerje višin ξ_p v trenutku preskoka lupine pa so zapisane v preglednici 1.

Kritična vrednost začetne oblikovne funkcije F_0 izpod katere preskok lupine ni mogoč, znaša za lupine s Poissonovim količnikom μ= 1/3, $F_0 = F_{kr}$ = 8,93. Krivulja s F_0 = 0 ponazarja razmere pri okrogli bimetalni plošči. Potek krivulje za brezrazsežno temperaturo τ v odvisnosti od razmerja višin ξ je asimetrična glede na premico F_0 = 0.

D(0.752,0.805) at the temperature τ= 0.805. By heating the shell up to the temperature of the first snap-through τ_{p1} = 1.195, the whole cycle of the shell's snaps is repeated.

The snap-through behaviour of the shells with different values of the initial formative function F_0 is shown in Fig. 6, while the tabulated values of the snap-through temperature τ_p and the ratio of height ξ_p at the moment of the shell's snap-through are presented in Table 1.

The critical value of the initial formative function F_0 under which the shell's snap-through is not possible amounts to $F_0 = F_{kr}$ = 8.93 for the shells with Poisson's ratio μ = 1/3.The curve with F_0 = 0 shows the conditions for the round bimetallic plate. The curve's line for dimensionless temperature τ relative to ratio ξ, is asymmetrical with respect to the straight line F_0 = 0.

Na sliki 7 sta prikazani krivulji za temperaturo prvega preskoka τ_{p1} in razmerje višin ξ_{p1} v odvisnosti od začetne oblikovne funkcije F_0.

Posledica temperaturne obremenitve prosto položene lupine je tudi vodoravni premik, določen z razliko med Eulerjevo in Lagrangevo koordinato *X-x*.

Enačba (48) za Eulerjevo koordinato *X* je, potem ko izrazimo komponento *u* vektorja premika $\vec{u}$ iz enačbe (52), specifično deformacijo ε_φ pa iz enačb (44) in (45):

Fig. 7 shows the curvatures for the temperature of the first snap-through τ_{p1} and the ratio of heights ξ_{p1} depending on the initial formative function F_0.

The consequence of the temperature loading of the simply roller-supported shell is also a horizontal displacement, determined by the difference between the Euler and Lagrange coordinates *X-x*.

After expressing the component *u* of the displacement vector $\vec{u}$ from Equation (52) and the normal strain ε_φ from Equations (44) and (45), Equation (48) for the Euler coordinate *X* is:

$$X = a\sqrt{\chi}\left(1 + \frac{n_\varphi - \mu n_\psi}{A\left(1-\mu^2\right)} + \frac{PT}{A(1+\mu)}\right) \quad (96).$$

Enotsko silo n_ψ izrazimo iz enačbe (67), enotsko silo n_φ pa z zvezo med obema enotskima silama iz enačbe (59). Vodoravni premik $X(\chi,\tau)$ je torej pri danih parametrih lupine odvisen od napetostne funkcije $G(\chi,\tau)$:

The unit force n_ψ is expressed by Equation (67) and the unit force n_φ by the connection between both unit forces by means of Equation (59). Hence, the horizontal displacement $X(\chi,\tau)$, for the given parameters of the shell, depends on the stress function $G(\chi,\tau)$:

$$X(\chi,\tau) = a\sqrt{\chi}\left(1 + \frac{C\left(2G'\chi + G(1-\mu)\right)}{a^2 A\left(1-\mu^2\right)} + \frac{P\cdot\tau\cdot T_M}{A(1+\mu)}\right) \quad (97).$$

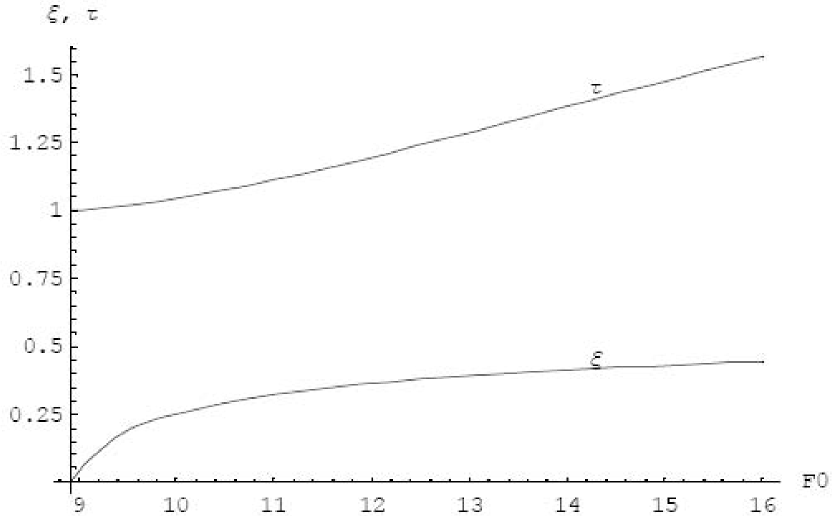
Za primer smo izračunali brezrazsežni vodoravni premik $(X(a) - a)/a$ v odvisnosti od temperature *t* za lupino s parametri :

As an example, we have calculated the dimensionless horizontal displacement $(X(a) - a)/a$ depending on the temperature τ for the shell with the parameters:

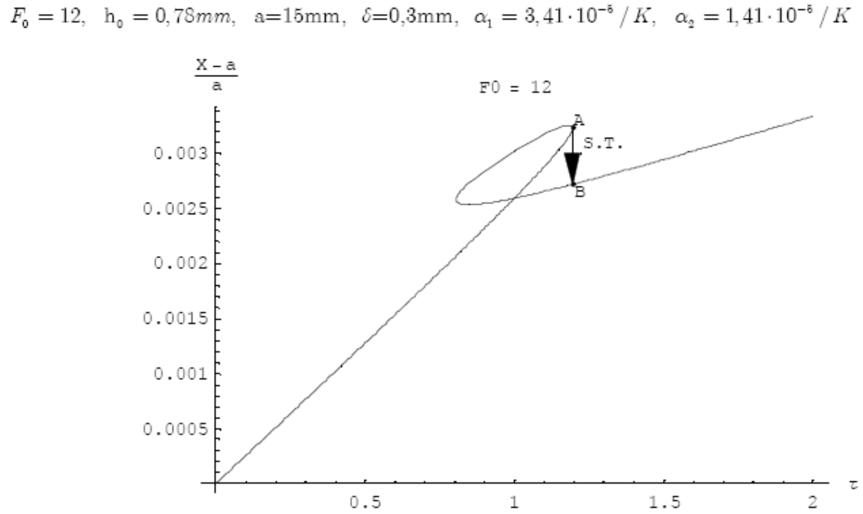
$$F_0 = 12,\quad h_0 = 0,78mm,\quad a=15\text{mm},\quad \delta=0,3\text{mm},\quad \alpha_1 = 3,41\cdot 10^{-5}/K,\quad \alpha_2 = 1,41\cdot 10^{-5}/K$$

Kakor je razvidno s slike 8, se vodoravni premik bimetalne lupine s temperaturo veča. Največji vodoravni premik je v trenutku preskoka v

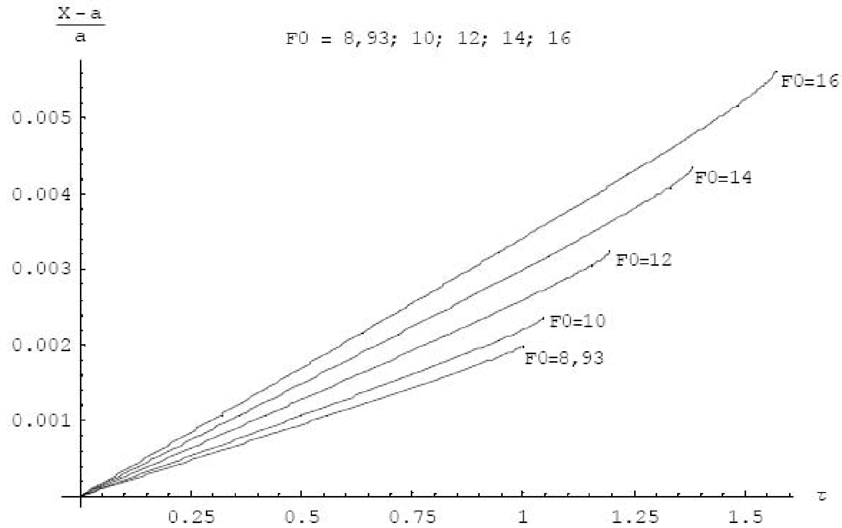
As shown in Fig. 8, the horizontal displacement of the bimetallic shell increases with temperature. The biggest horizontal displacement is at the beginning of



Sl. 7. *Temperatura in lega preskoka v odvisnosti od oblikovne funkcije F_0 za lupine z μ = 1/3*
Fig. 7. *Temperature and snap-through position depending on the formative function F_0 for the shells with μ = 1/3*



Sl. 8. *Vodoravni premik na robu lupine v odvisnosti od temperature t za prosto položeno lupino*
Fig. 8. *Horizontal displacement at the edge of the simply-roller supported shell relative to temperature τ*



Sl. 9. *Vodoravni premik na robu lupine za prosto položene lupine različnih vrednosti začetne oblikovne funkcije F_0*
Fig. 9. *Horizontal displacement at the edge of the simply-roller supported shell of various values of the initial formative function F_0*

točki *A*. Po preskoku zavzame lupina novo ravnovesno lego, vodoravni premik pa preide v točko *B*.

Na sliki 9 je prikazan vodoravni premik na robu lupine za lupine z debelino δ = 0,3 mm, tlorisnim polmerom a = 15 mm, razteznostnima koeficientoma $\alpha_1 = 3{,}41\cdot10^{-5}$ /K in $\alpha_2 = 1{,}41\cdot10^{-5}$ /K ter začetno oblikovno funkcijo F_0, kar izhaja s slike. S povečevanjem začetne višine h_0 in s tem

the snap-through process, at the point *A*. After the snap-through, the shell assumes a new equilibrium position, while the horizontal displacement passes to the point *B*.

Fig. 9 shows the horizontal displacement at the edge of the shell with thickness δ= 0.3 mm and horizontal radius a = 15 mm, coefficients of linear temperature expansion $\alpha_1 = 3.41\cdot10^{-5}$ /K and $\alpha_2 = 1.41\cdot10^{-5}$ /K and initial formative function F_0, as follows from the graphical presentation. By increasing the initial height h_0 and funkcije F_0 poleg temperature preskoka se zvečuje tudi vodoravni premik na robu lupine v $x = a$.

Obravnavali bomo tudi preskok sistema lupine z vrtljivim vodoravno nepomično vpetim robom. Takšen je primer, če lupino vstavimo v valj ter tako preprečimo širjenje lupine v smeri osi X. V tem primeru vpetja je premer lupine a med temperaturnim obremenjevanjem stalen: $a(T) = a$ = konst. Veljata robna pogoja: $\varepsilon_\varphi(1) = 0$; $m_\psi(1) = 0$.

Specifična deformacija ε_φ ima v primeru, ko sta debelini slojev enaki $\delta_1 = \delta_2 = \delta/2$ naslednjo obliko:

subsequently the function F_0, as well as an increase in the snap-through temperature the horizontal displacement at the edge of the shell at $x = a$ is increased too.

The next example we will discuss is the snap-through of the simply bearing-supported shell. This is the case when the shell is inserted into a cylinder, preventing in this way the expansion of the shell in the direction of the X axis. In this type of support, the radius of the shell a remains constant: $a(T) = a$ = const. during the temperature loading. The boundary conditions are: $\varepsilon_\varphi(1) = 0$; $m_\psi(1) = 0$.

The normal strain ε_φ has, in the case where the thicknesses of layers are equal, the following form:

$$\varepsilon_\varphi = \frac{\left(n_\varphi - \mu n_\psi\right) + PT\left(1-\mu\right)}{A\left(1-\mu^2\right)} \quad (98).$$

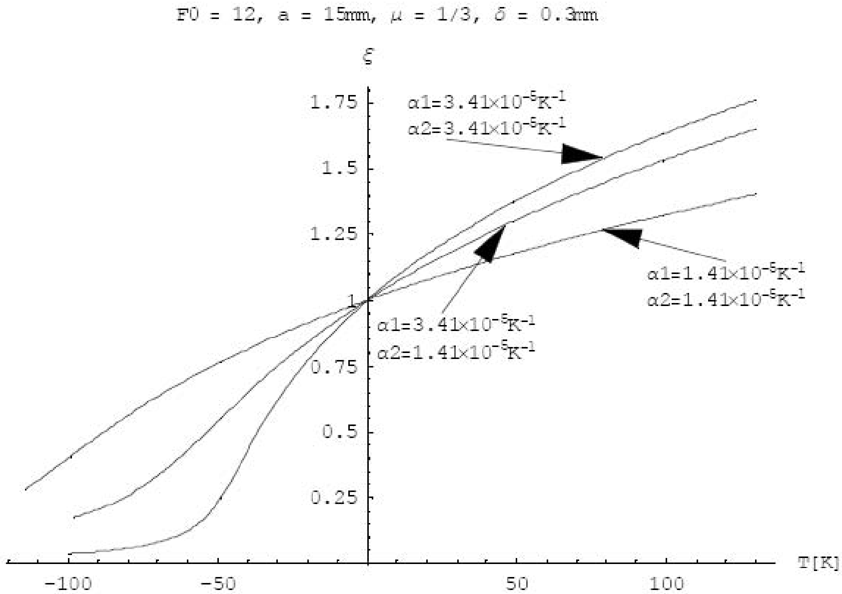
Ker se n_ψ in n_φ v (98) izražata z enačbama (67) in (59), zapišemo problem robnih vrednosti za vrtljivo vodoravno nepomično vpeto lupino:

Because n_ψ and n_φ in (98) are expressed by Equations (67) and (59) we can write the boundary-value problem for simply bearing-supported shells:

$$4g'' = F_0^2 - \frac{f^2}{\chi^2},\quad 4f'' = \frac{f\cdot g}{\chi^2}$$
$$g(0) = f(0) = 0,\quad g'(1) - g(1)\left(\frac{1+\mu}{2}\right) = \frac{-\sqrt{2/3}\cdot F_0\left(\alpha_1+\alpha_2\right)\sqrt{1-\mu^2}}{\alpha_1-\alpha_2}\cdot\tau \quad (99).$$
$$\tau = 1 - \frac{1}{F_0\left(1+\mu\right)}\left(2f'(1) - f(1)\cdot(1-\mu)\right)$$

Sistem enačb (99) smo rešili numerično s predhodno opisano nelinearno strelsko metodo.

The system of Equations (99) was solved numerically using the above-described non-linear



Sl. 10. *Razmerje višin ξ v odvisnosti od temperature T za vrtljivo vodoravno nepomično vpeto lupino*
Fig. 10. *Ratio of heights ξ relative to the temperature T for simply bearing-supported shell*

Izbrali smo približni začetni vrednosti za funkciji g in f v točki $\chi = 1$ ter problem začetnih vrednosti rešili po metodi Runge Kutta 4. reda. Bolj natančne funkcijske vrednosti $g(1)$ in $f(1)$ smo spet računali z Newtonovo metodo za reševanje nelinearnih enačb.

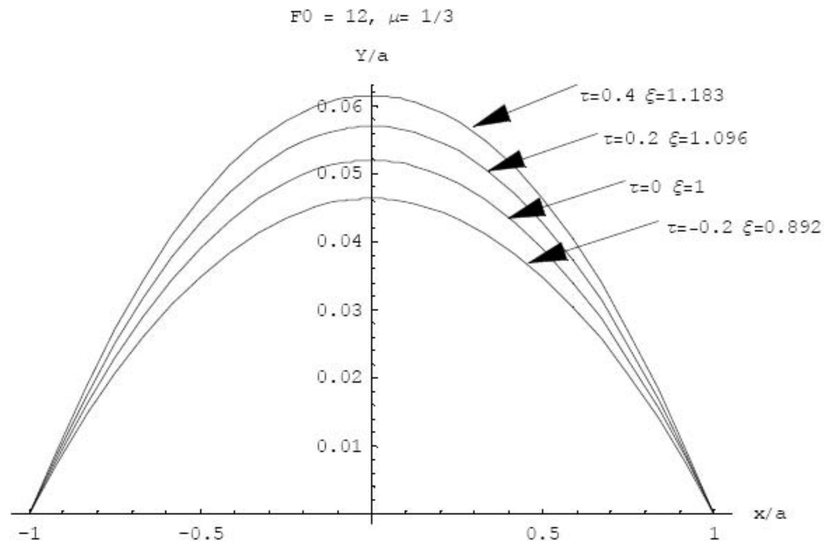
Na sliki 10 so predstavljene razmere pri temperaturnem obremenjevanju vrtljivo vodoravno nepomično vpete lupine za številčni primer $F_0 = 12$; $a = 15$ mm; $\mu = 1/3$; $\delta = 0{,}3$ mm.

Pri takšnem vpetju bimetalne lupine se razmerje višin ξ s povečevanjem temperature povečuje. Zaradi segrevanja se lupina razteza. Ker je raztezanje lupine v vodoravni smeri onemogočeno, se lupina razteza v smeri navpičnice. S temperaturo T se povečuje trenutna višina lupine Y in s tem razmerje ξ. Ker funkcija $\xi(T)$ nima lokalnega ekstrema sklepamo, da vrtljivo vpeta lupina za ta primer nima preskoka. Podobne razmere opažamo tudi pri enoslojnih lupinah oziroma lupinah z nespremenljivim koeficientom linearnega temperaturnega raztezka $\alpha_1 = \alpha_2$. Kakor je razvidno s slike 10, je deformacija bimetalne lupine s temperaturnim raztezkom slojev $\alpha_1 > \alpha_2$ nekje med deformacijama enoslojnih lupin z nespremenljivim temperaturnim raztezkom α_1 in α_2. Deformacijske krivulje, ki prikazujejo obliko lupine glede na temperaturno obremenitev, so razvidne s slike 11.

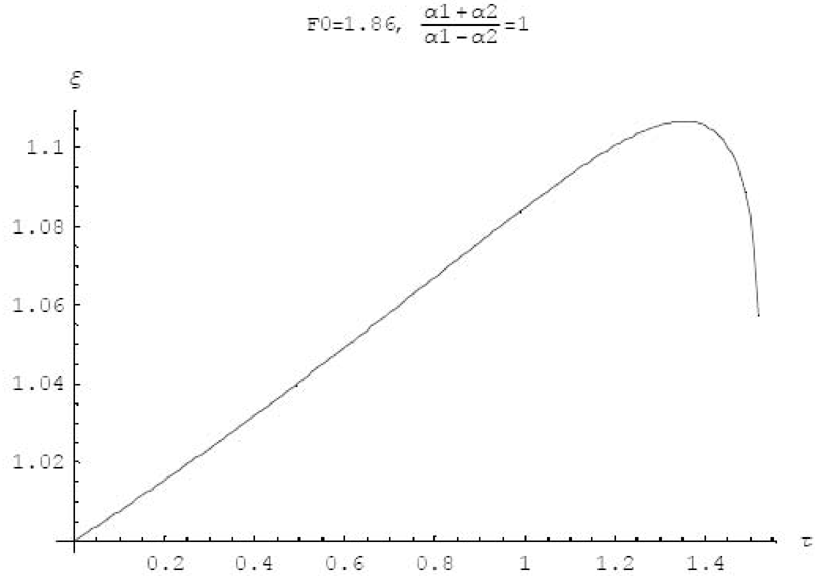
shooting method. The approximate initial values for the functions g and f at the point $\chi = 1$ were selected and the problem of the initial values solved by the Runge Kutta method of the fourth order. More precise functional values $g(1)$ and $f(1)$ were again calculated by means of Newton's method for solving non-linear equations.

Fig. 10 shows the conditions in the temperature loading of simply bearing-supported shells for the numerical sample $F_0 = 12$; $a = 15$ mm; $\mu = 1/3$; $\delta = 0.3$ mm.

In this example of the bimetallic shell, the ratio ξ increases with increasing temperature. Because of the heating, the shell is expanding. And since the expansion of a shell in the horizontal direction is not possible, the shell is expanding in the vertical direction. Along with the temperature T the height of the shell Y is also expanding, and with this the ratio ξ. Since the function $\xi(T)$ does not have a local extreme we can conclude that shells with a simple bearing-support do not have a snap-through. Similar results were also observed in the single-layer shells with a constant coefficient of linear temperature expansion $\alpha_1 = \alpha_2$. As seen from Fig. 10, the deformation of the bimetallic shell with a temperature expansion of layers $\alpha_1 > \alpha_2$ is somewhere between the deformations of single layer shells with constant temperature expansion α_1 and α_2. The deformation states of the shell relative to the temperature are shown in Fig. 11.

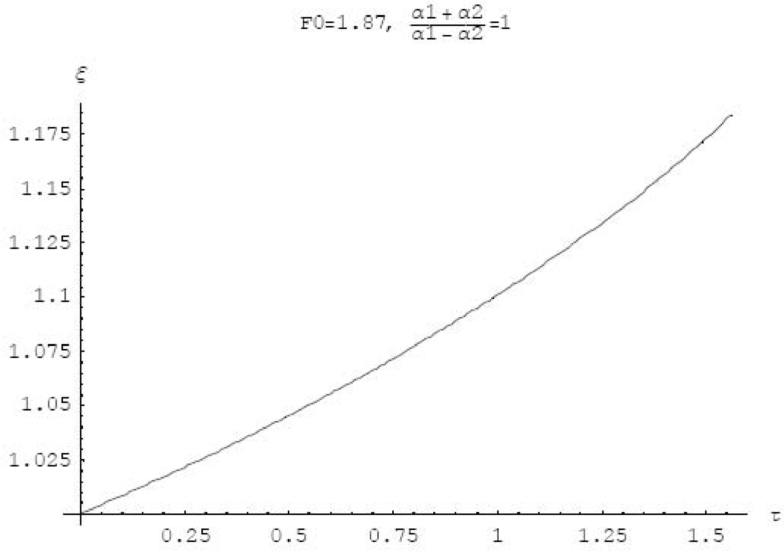


Sl. 11. *Deformacijske krivulje za vrtljivo vodoravno nepomično vpeto lupino s $F_0 = 12$ in $\mu = 1/3$*
Fig. 11. *Displacement states for simply bearing-supported shell with $F_0 = 12$ and $\mu = 1/3$*



Sl. 12. *Razmerje višin ξ v odvisnosti od temperature T za vrtljivo vodoravno nepomično vpeto lupino z $\mu = 1/3$, $(\alpha_1 + \alpha_2)/(\alpha_1 - \alpha_2) = 1$ in $F_0 = 1,86$*

Fig. 12. *Ratio of heights ξ relative to temperature T for simply bearing-supported shell with $\mu = 1/3$, $(\alpha_1 + \alpha_2)/(\alpha_1 - \alpha_2) = 1$ and $F_0 = 1.86$*



Sl. 13. *Razmerje višin ξ v odvisnosti od temperature T za vrtljivo vodoravno nepomično vpeto lupino z $\mu = 1/3$, $(\alpha_1 + \alpha_2)/(\alpha_1 - \alpha_2) = 1$ in $F_0 = 1,87$*

Fig. 13. *Ratio of heights ξ relative to the temperature T for simply bearing-supported shell with $\mu = 1/3$, $(\alpha_1 + \alpha_2)/(\alpha_1 - \alpha_2) = 1$ in $F_0 = 1.87$*

Analizirali smo tudi razmere pri zelo plitvih lupinah z majhno začetno višino h_0. Izkazalo se je, da se pri takšnih lupinah razmerje višin $\xi(T)$ z višanjem temperature T zmanjšuje, če je le začetna oblikovna funkcija F_0 dovolj majhna glede na

We have also analyzed the conditions in very shallow shells with a small initial height h_0. It turned out that in such shells the ratio $\xi(T)$ decreased with increasing temperature T only if the initial formative function F_0 was small enough compared to the ratio razmerje $(\alpha_1 + \alpha_2)/(\alpha_1 - \alpha_2)$ v sistemu enačb (99). Največja vrednost za začetno oblikovno funkcijo, pri kateri se razmerje višin $\xi(T)$ z višanjem temperature T še vedno zmanjšuje, znaša $F_0 = 1{,}86$ (sl. 12). Takrat ima razmerje med vsoto in razliko razteznostnih koeficientov najmanjšo mogočo vrednost $(\alpha_1 + \alpha_2)/(\alpha_1 - \alpha_2) = 1$.

Če vrednost začetne oblikovne funkcije F_0 le malenkostno povečamo, se bo razmerje višin $\xi(T)$ z višanjem temperature T zvečalo, kar je razvidno s slike 13.

V primeru konzolno vpete lupine sta robna pogoja: $\varepsilon_\varphi(1) = 0$; $Y'(1) = y'(1) =$ konst. Funkciji Y' in y' izrazimo iz enačb (69) in (68). Po krajšanju enakih členov je robni pogoj: $F(1) = F_0$. Problem robnih vrednosti za konzolno vpeto lupino je s tem:

$(\alpha_1 + \alpha_2)/(\alpha_1 - \alpha_2)$ in the system of equations (99). The highest value for the initial formative function, in which the ratio of heights $\xi(T)$ with increasing temperature T is still decreasing, is $F_0 = 1.86$, Fig. 12. In this case the relation between the sum and the difference of the expansion factor has the smallest possible value $(\alpha_1 + \alpha_2)/(\alpha_1 - \alpha_2) = 1$.

If we only slightly increase the value of initial formative function F_0, the ratio $\xi(T)$ will increase with the increase of the temperature T, as is clear from Fig. 13.

In the case of the clamped shell, the boundary conditions are: $\varepsilon_\varphi(1) = 0$; $Y'(1) = y'(1) =$ const. The functions Y' and y' are expressed from Equations (69) and (68). After reducing the same terms, the boundary condition is: $F(1) = F_0$. Then, the boundary-value problem for the clamped shell is:

$$4g'' = F_0^2 - \frac{f^2}{\chi^2}, \quad 4f'' = \frac{f \cdot g}{\chi^2}$$
$$g(0) = f(0) = 0, \quad f(1) = F_0, \quad g'(1) - g(1)\left(\frac{1+\mu}{2}\right) = \frac{-\sqrt{2/3} \cdot F_0 \left(\alpha_1 + \alpha_2\right)\sqrt{1-\mu^2}}{\alpha_1 - \alpha_2} \cdot \tau \qquad (100).$$

Tokrat smo izbrali približni vrednosti za $g(1)$ in $f'(1)$. Ob vsakem koraku smo preverili, kolikšno je odstopanje od robnih pogojev v točki $\chi = 0$ ter z Newtonovo metodo izračunali boljša približka za $g(1)$ in $f'(1)$. Rezultati so razvidni s slik 14 in 15 in so podobni razmeram pri vrtljivo vodoravno nepomično vpeti lupini.

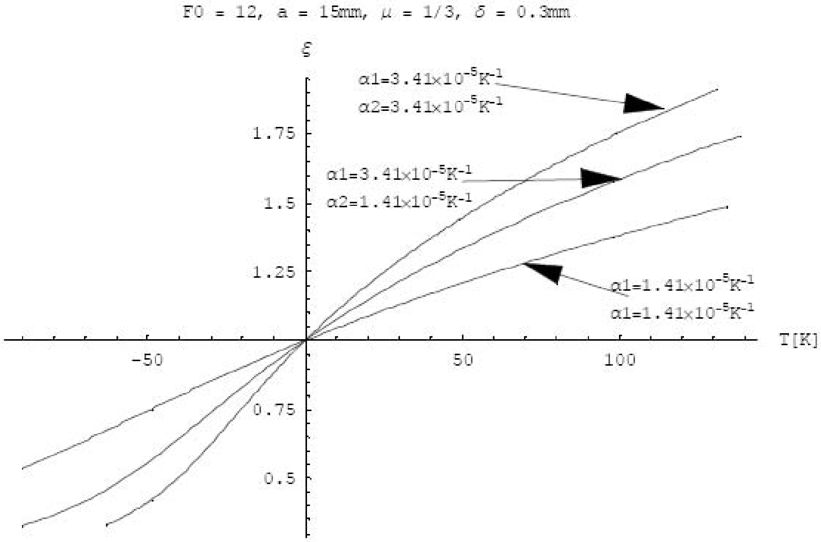
Razlika v stopnji deformacije različno vpetih lupin z enakimi snovno geometrijskimi značilkami je razvidna s slik 16 in 17. S C_0 smo označili obliko obeh lupin v nedeformiranem stanju, s C_1 obliko za vrtljivo vodoravno nepomično vpeto lupino in s C_2 obliko za konzolno vpeto lupino. Pri tem pomenita zgornji deformacijski krivulji obliko lupin pri temperaturi $\tau = 1$, spodnji krivulji pa obliko pri temperaturi $\tau = -1$. Kakor je razvidno s slike 16 je razlika v deformaciji različno vpetih lupin očitnejša šele pri večji temperaturni obremenitvi. Pri konzolno vpeti lupini se razmerje višin $\xi(T)$ z višanjem temperature T povečuje ne glede na vrednost začetne oblikovne funkcije F_0.

Oglejmo si še razmere pri analizi preskoka sistema pri prosto položeni lupini, ki je v temenu obremenjena z zunanjo silo $\vec{F}_k$. Ta obremenitveni primer opisujeta diferencialni enačbi (81) in (82). Iz primerjave enačb (74) in (75) izhaja, da je normalna enotska sila n_φ v primerjavi s strižno enotsko silo $\tau_{\psi r}$ majhna, zato jo zanemarimo: $n_\psi(1) \cong 0$.

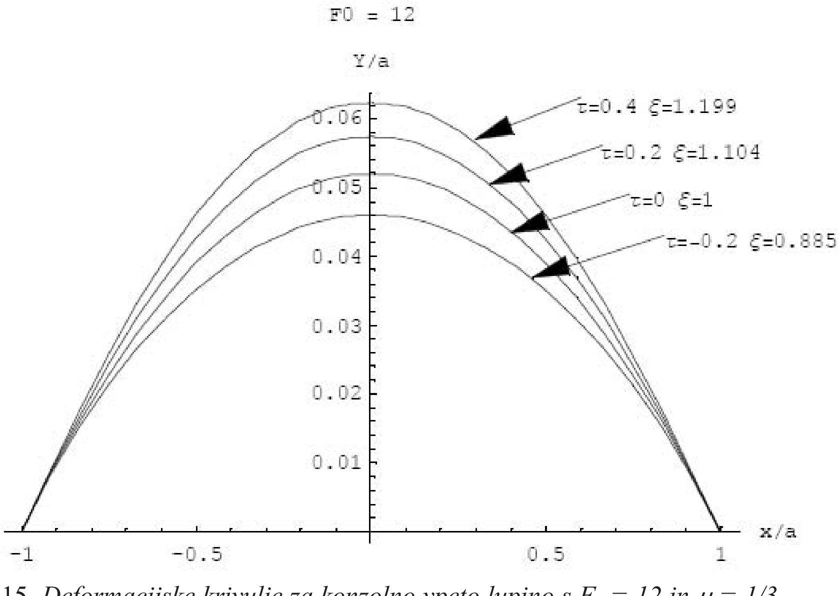
We chose approximate values for $g(1)$ and $f'(1)$. In each step we checked how far the digression from the boundary conditions can go at point $\chi = 0$ and by means of Newton's method, better proximities for $g(1)$ and $f'(1)$ are computed The results can be seen in Figures 14 and 15; they are similar to the conditions found in the simple bearing-supported shell.

Displacement states for different kinds of boundary conditions of the shells of the same material and geometrical characteristics can be seen in Figures 16 and 17. C_0 denotes the shape of both shells in the undeformed state, C_1 denotes the simple bearing-supported shell and C_2 the clamped shell. In such case the upper displacement curves represent the shell's shape at temperature $\tau = 1$ and the lower ones the shape at the temperature $\tau = -1$. As is clear from Figures 16 and 17, the difference in the displacement of the various boundary conditions of the shells is clearly expressed only in the case of higher temperature loads. In the clamped shells, the ratio of heights $\xi(T)$ is increasing with increasing temperature T, regardless of the value of the initial formative function F_0.

The object of discussion is also the snap-through phenomenon in the simple roller-supported shell, which is additionally loaded with an external force $\vec{F}_k$ in its top. This loading example is outlined by the differential equations (81) and (82). When comparing Equations (74) and (75) it follows that the normal unit force n_φ compared to the tangential force τ_ψ is small, hence it can be neglected: $n_\psi(1) \cong 0$.



Sl. 14. *Razmerje višin ξ v odvisnosti od temperature T za konzolno vpeto lupino*
Fig. 14. *Ratio of heights ξ relative to the temperature T for clamped shell*

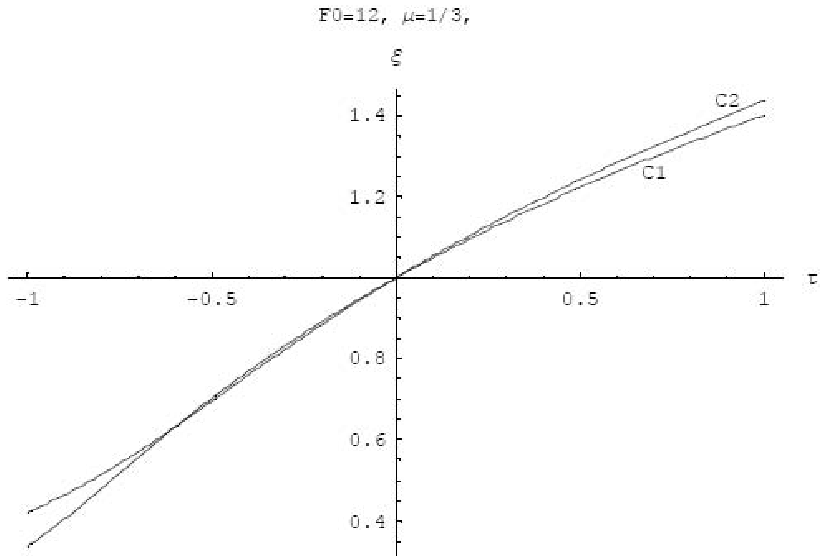


Sl. 15. *Deformacijske krivulje za konzolno vpeto lupino s F_0 = 12 in μ = 1/3*
Fig. 15. *Displacement states for clamped shell with F_0 = 12 and μ = 1/3*

Če ponovno upoštevamo, da sta funkciji G in F v točki $\chi = 0$ omejeni, zapišemo problem robnih vrednosti:

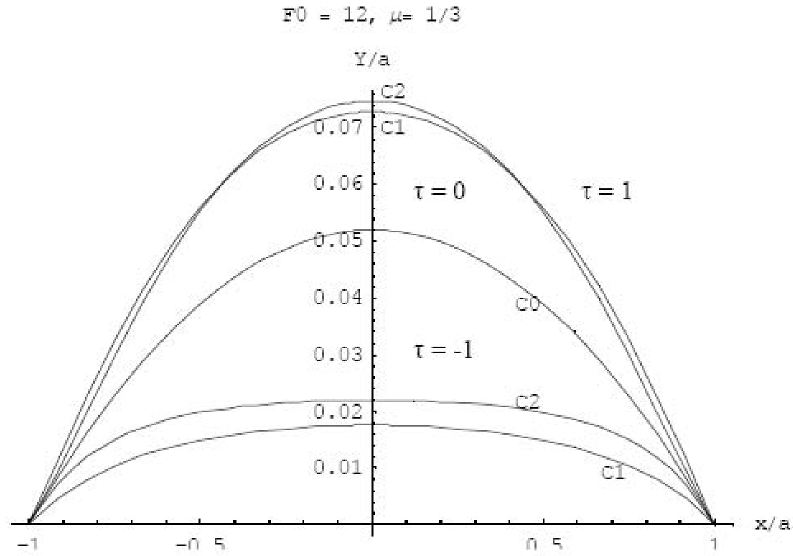
Considering that functions G and F at point $\chi = 0$ are limited, the boundary-value problem can be expressed as:

$$4g'' = F_0^2 - \frac{f^2}{\chi^2},\quad 4f'' = \frac{f \cdot g}{\chi^2} + \frac{f_k}{\chi}$$
$$g(0) = g(1) = f(0) = 0,\quad \tau = 1 - \frac{1}{F_0(1+\mu)}\left(2f'(1) - f(1)\cdot(1-\mu)\right) \qquad (101),$$



Sl. 16. *Razmerje višin ξ pri vrtljivo vodoravno nepomično vpeti lupini (C_1) in konzolno vpeti lupini (C_2) v odvisnosti od temperaturnega obremenjevanja za lupine s F_0 = 12 in μ = 1/3*

Fig. 16. *Ratio of heights ξ in simply bearing-supported shell (C_1) and clamped shell (C_2) depending on temperature loading for shells with F_0 = 12 and μ = 1/3*



Sl. 17. *Deformacijske krivulje pri vrtljivo vodoravno nepomično vpeti lupini (C_1) in konzolno vpeti lupini (C_2) za temperaturne vrednosti τ = 0, τ = -1, τ = 1*

Fig. 17. *Displacement states in simply bearing-supported shell (C_1) and clamped supported shell (C_2) for . temperature values τ = 0, τ = -1, τ = 1*

kjer je f_k, po analogiji z brezrazsežnotemperaturo τ, brezrazsežna sila, F_M pa nespremenljiva:

where f_k is a dimensionless force, and F_M is a constant:

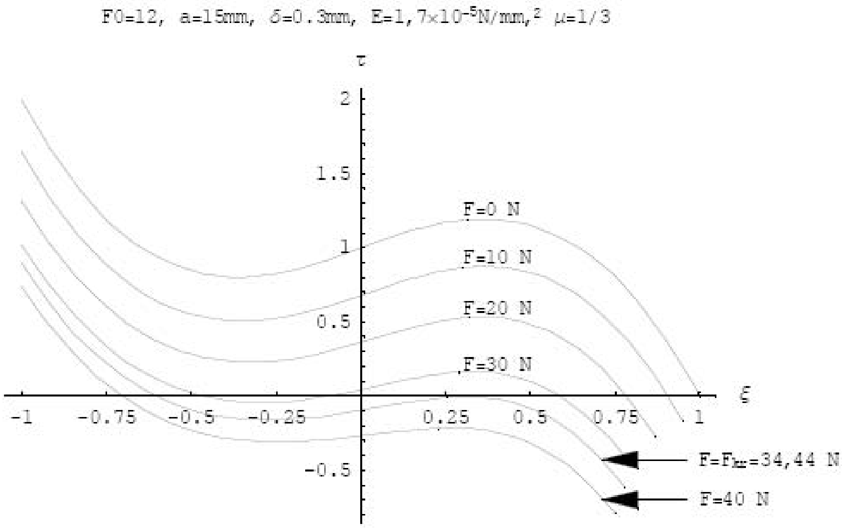
$$F_k = f_k \cdot F_M; \quad F_M = \frac{2\pi}{a^2}\sqrt{\frac{2C^3}{A\left(1-\mu^2\right)}} = \frac{E\,\pi\,\delta^4}{a^2 6\sqrt{6\left(1-\mu^2\right)^3}}$$

Reševanje sistema enačb (101) je potekalo po prej opisani numerični metodi. Zaradi nazornosti

To solve the system of Equations we use the above-described numerical method. For the sake of

Preglednica 2. *Temperatura in lega preskoka lupine za različne vrednosti sile F_k*
Table 2. *Temperature and shell's snap-through position for different force values F_k*

$F_k\,[N]$	$0N$	$10N$	$20N$	$30N$	$F_{kr}=34,44N$	$40N$
$S.T.1$	$\tau_{p1}=1,195$ $\xi_1=0,361$	$\tau_{p1}=0.879$ $\xi_{p1}=0.366$	$\tau_{p1}=0.535$ $\xi_{p1}=0,360$	$\tau_{p1}=0.166$ $\xi_{p1}=0,335$	$\tau_{p1}=0$ $\xi_{p1}=0,312$	$\tau_{p1}=-0,214$ $\xi_{p1}=0,269$
$S.T.2$	$\tau_{p2}=0,805$ $\xi_{p2}=-0,361$	$\tau_{p2}=0.510$ $\xi_{p2}=-0.348$	$\tau_{p2}=0.231$ $\xi_{p2}=-0,327$	$\tau_{p2}=-0,040$ $\xi_{p2}=-0,293$	$\tau_{p2}=-0.159$ $\xi_{p2}=-0,274$	$\tau_{p2}=-0,308$ $\xi_{p2}=-0,240$



Sl. 18. *Stabilnostne razmere pri različnih vrednostih sile F_k*
Fig. 18. *Snap-through behaviour for different force values F_k*

predstavljamo grafične in pregledničneo rezultate za lupino z naslednjimi podatki:

clarity, the graphical and tabular results for the shell are shown below, with the following data:

$$F_0=12,\quad h_0=0,78mm,\quad a=15mm,\quad \delta=0.3mm,\quad E=1,7\cdot 10^5\,MPa \qquad (102).$$

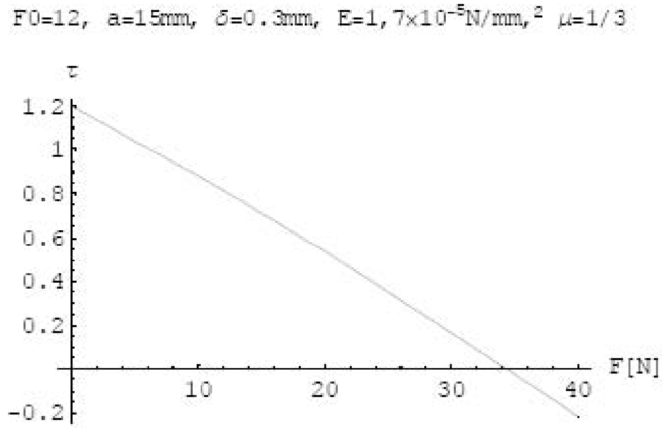
S slike 18 je razvidno, da se temperatura preskoka lupine τ_p znižuje z večanjem sile F_k. Pri sili F_k = 0 smo za temperaturo τ_p in lego ξ_p preskoka lupine dobili enake rezultate, kakor smo jih predhodno izračunali za samo temperaturno obremenjeno lupino. Temperaturo preskoka τ_{p1} za vmesne vrednosti sile F_k v območju [0,40] smo določili z interpolacijskim polinomom 4. stopnje (sl. 19):

From Fig.18 it follows that the snap-through temperature τ_p is decreasing with increasing force F_k. For the force F_k = 0 we get the same results for the temperature τ_p and position ξ_p of shell's snap-through as those previously calculated for the shell that was loaded only by temperature. The snap-through temperature τ_{p1} for the intermediate values of the force F_k in the interval [0,40] was determined by the interpolating polynomial of the fourth degree, Fig. 19:

$$\tau_{p1}\left(F_k\right)=1,195-3,03\cdot 10^{-2}F_k-1,045\cdot 10^{-4}F_k^2-2,25\cdot 10^{-6}F_k^3+4,583\cdot 10^{-8}F_k^4 \qquad (103).$$

S polinomom (103) smo določili kritično silo F_{kr}, pri kateri lupina preskoči pri temperaturi $\tau=0$:

Using the polynomial (103) the critical force F_{kr}, at which the shell snaps-through at temperature $\tau=0$, is determined:



Sl. 19. *Temperatura preskoka* τ_{p1} *v odvisnosti od zunanje sile* $\vec{F}_k$

Fig. 19. *Snap-through temperature* τ_{p1} *depending on the external force* $\vec{F}_k$

Pri kritični sili F_{kr} = 34,44 N lupina preskoči, ne da bi jo bilo potrebno segrevati. Če se po preskoku lupine sila zmanjša, se zmanjša tudi razmerje višin ξ. Kadar je lupina dovolj plitva, se po prenehanju sile vrne v izhodiščno lego. Obravnavana lupina je že takšna, saj je pri mehansko neobremenjeni lupini $F_k = 0$ možno samo eno ravnovesno stanje pri temperaturi $\tau = 0$ in sicer pri razmerju višin $\xi = 1$. Da bi lupina po prenehanju kritične sile F_{kr} ne preskočila v izhodiščno lego, mora krivulja za temperaturo τ v odvisnosti od razmerja višin ξ sekati negativni del abscisne osi ali se je vsaj dotakne. Samo v tem primeru sta mogoči dve stabilni ravnovesni stanji in razmerje višin $\xi \neq 1$ pri temperaturi $\tau = 0$. Približno najmanjšo vrednost začetne oblikovne funkcije $F_{0\min}$ izračunamo z interpolacijskim polinomom. Interpoliramo preglednico 1, in sicer temperaturo τ_{p2} drugega preskoka (S.T.2) z začetno oblikovno funkcijo F_0 v območju $[8,93 \leq F_0 \leq 16]$:

At the critical force F_{kr} = 34.44 N the shell snaps-through without being heated up. If after the shell's snap-through the force decreases, the ratio ξ is decreased too. When the shell is shallow enough, on removal of the force it returns to the initial position. The shell we are discussing is of this type, because only one equilibrium state is possible at temperature $\tau = 0$ at $\xi = 1$ if there is no external force in the top of the shell. To prevent the shell snapping-through into the initial position on removal of the critical force F_{kr} the curve for the temperature τ depending on the ratio ξ should be crossing the negative part of the abscissa axis or at least touch it. Only in this way are two stable equilibrium conditions and a ratio of heights $\xi \neq 1$ possible at the temperature $\tau = 0$. As an approximation, the lowest value of the initial formative function $F_{0\min}$ is calculated using the interpolating polynomial. In Table 2 we interpolate the temperature τ_{p2} of the second snap-through (S.T.2) with the initial formative function F_0 in the interval $[8,93 \leq F_0 \leq 16]$:

$$\tau_{p2}(F_0) = -2,85 + 1,23F_0 - 1,36 \cdot 10^{-1} F_0^2 + 6,26 \cdot 10^{-3} F_0^3 - 1,07 \cdot 10^{-4} F_0^4$$

Postavimo:

Thus:

$$\tau_{p2}(F_{0\min}) = 0 \Rightarrow F_{0\min} \cong 19,96$$

Ekstrapolirana vrednost za $F_{0\min}$ je seveda približna, ker leži zunaj interpolacijskega območja [8,93;16]. Vseeno je dober približek za numerični izračun temperature preskoka τ_{p2}. V drugi iteraciji že dobimo praktično dovolj natančno vrednost, $F_{0\min}$ = 21,66 s temperaturo 2. preskoka τ_{p2} = -0,0001 in razmerjem ξ_{p2} = -0,530.

The extrapolated value $F_{0\min}$ is an approximate value, since it lies outside the interpolation interval [8.93,16]. Nevertheless, it is a good approximation for the numerical calculation of the temperature τ_{p2} snap-through. In the second iteration we already obtain a sufficiently accurate value, $F_{0\min}$ = 21.66 with the temperature of the second snap-through τ_{p2} = -0.0001 and the ratio ξ_{p2} = -0.530.

6 SKLEP

Prosto položene tankostenske plitve bimetalne lupine imajo lastnost, da pri določeni temperaturi preskočijo v novo ravnovesno lego. Temperatura preskoka T_p je odvisna od snovno geometrijskih lastnosti lupin. Kot poseben primer smo analizirali razmere pri sferičnih lupinah, ki imajo oba sloja enako debela $\delta_1 = \delta_2 = \delta/2$, sloja pa imata tudi enak Poissonov koeficient $\mu_1 = \mu_2 = \mu$. Iz enačbe (89) izhaja, da je lega preskoka ξ_p, pri lupinah z enako debelino δ in Poissonovim številom μ, odvisna samo od začetne višine lupine h_0. Ukrivljenost lupine $1/R$ in razlika temperaturnih koeficientov linearnega raztezka $\alpha_1 - \alpha_2$ vplivata samo na temperaturo preskoka T_p, ne pa tudi na lego preskoka ξ_p. Pri plitvih enoslojnih lupinah z nespremenljivim koeficientom linearnega temperaturnega raztezka $\alpha(z) = \alpha =$ konst ostaja razmerje višin $\xi(T)$ stalno ne glede na temperaturno obremenitev. Z višanjem temperature T se veča vodoravni premer lupine a, medtem ko ostaja navpična komponenta deformacije Y na robu lupine ves čas enaka:

6 CONCLUSION

In thin-walled, shallow bimetallic shells a snap-through into a new equilibrium state occurs when a certain temperature is reached. The snap-through temperature T_p depends on the material and the geometrical properties of the shell. As a special case, the stability conditions for spherical shells whose two layers have equal thickness $\delta_1 = \delta_2 = \delta/2$ and the same Poisson's ratio $\mu_1 = \mu_2 = \mu$ were analyzed. From Equation (89) it follows that the position of a snap-through ξ_p, in shells with equal thickness δ and Poisson's number μ, depends only on the initial value of the shell's height h_0. The curvature of the shell $1/R$ and the difference in the coefficients of the linear expansion $\alpha_1 - \alpha_2$ affect only the snap-through temperature T_p and have no influence on the snap-through ξ_p position. In shallow, single-layer shells with a constant coefficient of the linear temperature expansion $\alpha(z) = \alpha =$ const. the ratio of heights $\xi(T)$ remains constant, regardless of the temperature loading. With increasing temperature T the shell's horizontal radius a increases while the vertical component of the deformation Y at the edge of the shell always remains the same:

$$Y(a) = y(a) = \text{konst} \qquad (104).$$

Iz enačbe (104) izhaja, da enoslojne lupine nimajo preskoka. Prav tako nimajo preskoka plitve bimetalne lupine, ki imajo vrednost začetne oblikovne funkcije $F_0 < 8{,}93$, če znaša Poissonov koeficient $\mu = 1/3$. Da se pojavi pri bimetalnih lupinah preskok, je treba poleg dovolj visoke temperature zagotoviti, da se rob lupine lahko v vodoravni smeri prosto razteza. Največji vodoravni premik nastane na robu lupine v trenutku preskoka lupine. Bimetalne lupine z večjo začetno višino h_0 imajo ob enakih snovno geometrijskih značilkah večji vodoravni premik.

Vpete bimetalne lupine se v vodoravni smeri ne morejo raztezati. Ker je raztezanje takšne lupine v vodoravni smeri onemogočeno, se lupina razteza v smeri navpičnice, zaradi česar se razmerje višin $\xi(T)$ s povečevanjem temperature povečuje, $\xi(T) > 1$ za $T < 0$, razen pri vrtljivo vodoravno nepomično vpeti lupini z majhno začetno oblikovno funkcijo F_0.

Če na lupino poleg spremembe temperature T deluje tudi sila F_k v temenu lupine, se pojavi preskok lupine pri nižji temperaturi τ_p. Pri dovolj veliki sili F_{kr} lupina preskoči, ne da bi jo bilo treba dodatno segrevati. Velikost kritične sile F_{kr} je odvisna od snovno geometrijskih lastnosti lupine v enačbah

From Equation (104) it follows that single-layer shells have no snap-through. Also, very thin bimetallic shells with the initial formative function $F_0 < 8.93$ and Poisson's ratio $\mu = 1/3$ have no snap-through. In order that snap-through can occur in bimetallic shells, it is necessary, besides a high enough temperature, to ensure that the edge of the shell can freely expand in the horizontal direction. The greatest horizontal displacement occurs at the edge of a shell at the moment of the shell's snap-through. Bimetallic shells with a greater initial height h_0 have, for the same material and geometrical characteristics, a greater horizontal displacement.

Clamped bimetallic shells cannot expand in the horizontal direction. Because the expansion in the horizontal direction of such shell is prevented, the shell is expanding vertically. Consequently, the ratio of heights $\xi(T)$ increases along with the temperature increase $\xi(T) > 1$ for $T < 0$, except in simple bearing-supported shells with a small initial formative function F_0.

When in addition to the temperature T, also a force F_k is acting on the shell's top, the snap-through occurs at a lower temperature τ_p. When the force F_{kr} is high enough, the shell snaps-through without the need to be additionally loaded with temperature. The value of the critical force F_{kr} depends on the material and the geometrical

(102). Po prenehanju kritične sile se lupina vrne v začetno ravnovesno lego, razen pri manj plitvih lupinah, pri katerih je vrednost začetne oblikovne funkcije $F_{0\min} \geq 21,66$ za lupine z značilkami v enačbah (102).

properties of the shell in Equations (102). When the force is removed, the shell returns to its initial equilibrium state, except in the case of shallower shells, whose value of initial formative function is $F_{0\min} \geq 21,66$ for shells with the characteristics in Equations (102).

7 OZNAKE
7 SYMBOLS

tlorisni polmer nedeformirane lupine	a	horizontal radius of undeformed shell
Youngov elastièni modul materiala 1 in 2	E_1, E_2	Young's elastic modulus of materials 1 and 2
zaèetna oblikovna nedeformirana funkcija	$F_0(\chi)$	initial formative function
trenutna oblikovna deformirana funkcija	$F(\chi)$	present formative function
sila v temenu lupine	F_k	force acting at the top of a shell
kritièna sila v temenu lupine	F_{kr}	critical force acting at the top of a shell
èleni potenène vrste trenutne oblikovne funkcije	f_i	elements of power series of the present formative function
napetostna funkcija	$G(\chi)$	stress function
èleni potenène vrste trenutne napetostne funkcije	g_i	elements of power series of the stress function
zaèetna višina nedeformirane lupine	h_0	initial height of the undeformed shell
razdalji osrednje ploskve od spodnje (h_1) oziroma zgornje (h_2) ploskve bimetalne lupine	h_1, h_2	middle plane distance from the lower (h_1) and upper (h_2) plane of the bimetallic shell
ukrivljenosti nedeformirane lupine	k_ψ, k_φ	the curvatures of the undeformed shell
ukrivljenosti deformirane lupine	$\bar{k}_\psi, \bar{k}_\varphi$	the curvatures of the deformed shell
notranja momenta v lupini	M_ψ, M_φ	internal moments in the shell
notranja enotska momenta v lupini	m_ψ, m_φ	internal unit moments in the shell
notranji sili v smeri normale v lupini	N_ψ, N_φ	internal forces in the direction normal to the shell
notranji enotski sili v smeri normale na lupino	n_ψ, n_φ	internal unit forces in the direction normal to the shell
polmer nedeformirane krogelne lupine	R	radius of the undeformed spherical shell
polmera ukrivljenosti nedeformirane lupine	r_ψ, r_φ	radii of the undeformed shell
polmera ukrivljenosti deformirane lupine	$\bar{r}_\psi, \bar{r}_\varphi$	radii of the deformed shell
dol□ina na nedeformirani lupini	s	the length on the undeformed shell
dol□ina na deformirani lupini	$\bar{s}$	the length on the deformed shell
primerjalna temperatura	T_0	reference temperature
temperatura, temperatura preskoka	T, T_p	temperature, snap-through temperature
notranja stri□na sila na lupini	$T_{\psi r}$	internal tangential force in the shell
notranja enotska stri□na sila v lupini	$t_{\psi r}$	internal unit tangential force in the shell
vektor premika	$\vec{u}$	displacement vector
èleni vektorja premika	u, v, w	elements of the displacement vector
sila podpore, enotska sila podpore	V, V_e	reaction force, unit reaction force
Lagrangev koordinatni sistem	(x, y, z)	Lagrange coordinates
Eulerjev koordinatni sistem	(X, Y, Z)	Euler coordinates
temperaturna koeficienta dol□inskega raztezka materiala 1 in 2	α_1, α_2	linear temperature expansion coefficients of material 1 and material 2

debelina lupine	δ	shell's thickness
debelini slojev iz materiala 1 in 2	δ_1, δ_2	thickness of layers made of material 1 and 2
deformacijski tenzor	ε_{ij}	strain tensor
Poissonovi števili za material 1 in 2	μ_1, μ_2	Poisson's ratios of materials 1 and 2
napetostni tenzor	σ_{ij}	stress tensor
brezrazsežna temperatura, temperatura preskoka	τ, τ_p	dimensionless temperature, snap-through temperature
brezrazsežna neodvisna spremenljivka	χ	dimensionless independent variable
razliki ukrivljenosti	$\Upsilon_\psi, \Upsilon_\varphi$	curvature differences
kota na nedeformirani lupini	ψ, φ	angles of the undeformed shell
kot na deformirani lupini	$\bar{\psi}$	angles of the deformed shell

8 LITERATURA
8 REFERENCES

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