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Analiza toka newtonskih in nenewtonskih tekočin s tokovno funkcijo in vrtinčnostjo

Newtonian and Non-Newtonian Fluid Flow Analysis Using the Stream Function and Vorticity

Jure Marn - Marjan Delić

Analizirana je primernost formualcije ψ - ω metode končnih razlik za izračun tokovnih razmer v nestisljivi viskozni newtonski tekočini in podane so smernice za izračun razmer v nenewtonski tekočini. Prav tako je analizirana natančnost numeričnih rezultatov v odvisnosti od gostote mreže. Metoda je za primer newtonske tekočine testirana na primeru gnane kotanje, numerično dobljene vrednosti pa so primerjane z rezultati iz literature.

(Ključne besede: tekočine nestisljive, analize toka, metode Newton-a, metode končnih razlik)

The authors have analyzed the suitability of the ψ - ω formulation of the finite difference method to calculate incompressible viscous newtonian fluid flow as well as to assess the guidelines in order to complete the calculations for non-newtonian flows. In addition, convergence criteria are presented, and convergence depending on mesh size is analyzed. The method for newtonian flows is tested in the driven cavity setup and compared to results available in open literature.

(Keywords: incompressible fluids, flow analysis, Newton's method, finite diference methods)

0 UVOD

V prispevku je prikazana formulacija ψ - ω metode končnih razlik [1], [4] in [6] za računanje tokovnih razmer v viskozni nestisljivi newtonski in nenewtonski tekočini. Formulacija je za newtonske tekočine testirana na primeru gnane kotanje, kjer že pri majhnem Reynoldsovem številu nastajajo dodatni vrtinci, ki še povečajo zahtevnost numeričnega izračuna. Problem, ki ga predstavlja tlačni člen pri reševanju Navier-Stokesovih enačb, lahko odpravimo, če gibalno enačbo prevedemo v vrtinčno enačbo, saj tako iz enačbe izločimo tlak, katerega porazdelitev lahko pozneje izračunamo iz gibalne enačbe. Prav tako je primerno hitrostno polje nadomestiti s tokovno funkcijo, saj tako dosežemo tudi neintegralskimi metodami boljšo konvergenco ob upoštevanju težav zavoljo izračuna robnih vrednosti vrtinčnosti. Z vpeljavo tokovne funkcije in vrtinčnosti prav tako prevedemo trienačbni model v dvoenačbnega, preostala fizikalna polja pa lahko izračunamo pozneje iz znanih vrednosti vrtinčnosti in tokovne funkcije. V primeru preprostih newtonskih modelov je napetostni tenzor dokaj lahko izračunljiv, pri zahtevnejših modelih, kakršen je Oldroyd-B, pa moramo reševati dodaten vezan sistem diferencialnih enačb.

0 INTRODUCTION

This paper deals with ψ - ω formulation of finite differences [1], [4] and [6] for assessment of newtonian and non-newtonian incompressible fluid flow. For newtonian flows, the formulation is tested in a driven cavity setup, characterized by small vortex formation at low Reynolds number flow, these vortices adding to the numerical difficulties. The problem presented by the behavior of pressure term in the Navier-Stokes equation can be alleviated by applying curl to the momentum conservation equation, thus obtaining a vorticity transport equation. The pressure distribution can later be evaluated explicitly using an unhindered version of the momentum equation, should the information be necessary. In addition, the velocity field was replaced by streamfunction, thereby achieving better convergence rate by using non-integral methods taking into account problems in calculating the boundary values of vorticity. By the implementation of streamfunction, the three equation model is reduced to a two-equation model, while the remaining field properties can be evaluated directly from corresponding conservation equations. A similar mechanism can be used for simpler models of non-newtonian flows, while more sophisticated approaches (e.g. Oldroyd-B) require the solution of a coupled system of differential equations.

1 DEFINICIJA PROBLEMA

Ravninski tok viskozne nestisljive tekočine lahko opišemo z gibalnima enačbama in kontinuitetno enačbo, ki jih ob zanemaritvi prostorninskih sil zapišemo v obliki:

1 DEFINITION OF THE PROBLEM

The 2D flow of viscous incompressible flow can be described by momentum conservation and continuity (mass conservation) equation, neglecting volumetric forces, as follows:

$$\rho\left[\frac{\partial\vec{\upsilon}}{\partial t}+\left(\vec{\upsilon}\cdot\vec{\nabla}\right)\vec{\upsilon}\right]=-\vec{\nabla}p+\vec{\nabla}\cdot\underline{\tau} \qquad (1),$$

$$\vec{\nabla}\cdot\vec{\upsilon}=0 \qquad (2).$$

Za newtonske tekočine, pri katerih so strižne napetosti sorazmerne odvodu hitrosti, lahko zapišemo:

For newtonian fluids using Newton's viscous law where shear stress is proportional to the velocity derivative the following is written:

$$\rho\left[\frac{\partial\vec{\upsilon}}{\partial t}+\left(\vec{\upsilon}\cdot\vec{\nabla}\right)\vec{\upsilon}\right]=-\vec{\nabla}p+\nu\vec{\nabla}\vec{\upsilon} \qquad (3).$$

Zaradi računskih težav s tlačnim gradientom v gibalni enačbi je upravičeno gibalno enačbo spremeniti v vrtinčno obliko [8]:

As stated previously, problems arising from pressure term behavior during numerical calculations can be alleviated by applying the curl to the momentum conservation equation [8]:

$$\frac{\partial\omega}{\partial t}+\left(\vec{\upsilon}\cdot\vec{\nabla}\right)\omega=\frac{1}{\rho}\vec{\nabla}\times\vec{\nabla}\cdot\underline{\tau} \qquad (4).$$

Z vpeljavo tokovne funkcije ψ in vrtinčnosti ω prevedemo trienačbni model v dvoenačbnega. Z upoštevanjem definicijskih enačb za tokovno funkcijo in vrtinčnost:

By previously/ first using the streamfunction ψ and vorticity ω the three-equation model is transformed to a two-equation model. Using the definition for streamfunction and vorticity

$$\upsilon_{\chi}=\frac{\partial\psi}{\partial y} \qquad \upsilon_y=-\frac{\partial\psi}{\partial x} \qquad \omega=\frac{\partial\upsilon_y}{\partial x}-\frac{\partial\upsilon_x}{\partial y} \qquad (5),$$

kontinuitetno enačbo (2) zapišemo v obliki Poissonove enačbe za ψ:

the continuity (2) can be written in Poisson equation form for ψ:

$$\Delta\psi+\omega=0 \qquad (6)$$

za robne pogoje:

$$\psi=\overline{\psi} \quad \text{na} \quad \Gamma$$
$$\frac{\partial\psi}{\partial n}=\overline{\upsilon_t} \quad \text{na} \quad \Gamma \qquad (7),$$

kjer je z Γ označen rob območja.

Enačbo (6) uporabimo za izračun robnih vrednosti vrtinčnosti:

$$\omega_i=-\Delta\psi_r \quad \text{na} \quad \Gamma \qquad (8)$$

in tokovne funkcije v območju ψ_Ω, oziroma prek definicijske enačbe (5) določimo hitrostno polje.

Viri [2], [3] in [9] navaja več izrazov za izračun robnih vrednosti vrtinčnosti, kakor je na primer izraz Orszaga in Israelija [9]:

with boundary values

$$\psi=\overline{\psi} \quad \text{on} \quad \Gamma$$
$$\frac{\partial\psi}{\partial n}=\overline{\upsilon_t} \quad \text{on} \quad \Gamma \qquad (7),$$

where by Γ denotes the domain boundary.

Equation (6) can be used for boundary value calculation of the vorticity

$$\omega_i=-\Delta\psi_r \quad \text{on} \quad \Gamma \qquad (8)$$

and the streamfunction in the domain ψ_Ω, or the velocity field, is determined using the definition (5).

The references [2], [3] and [9] suggest several expressions for boundary values calculations such as Orszag and Israeli [9]:

$$\omega_i=\frac{1}{3(\Delta x)^2}\left(10\psi_{i+1}-\psi_{i+2}\right) \qquad (9).$$

Alternativno bi bilo mogoče problem izračuna robnih vrednosti uspešno odpraviti v

Alternatively, the boundary calculation problem could be succesfully overcome in integral formuintegralski robni formulaciji (metoda robnih elementov), pri kateri so robne vrednosti vsebovane v sami formulaciji, zaradi česar ne potrebujemo dodatnih izrazov za izračun vrtinčnosti na robu, kar ugodno vpliva na stabilnost metode [5] in [8].

Kinetiko toka podamo s prenosno enačbo (4) vrtinčnosti, ki jo za newtonske tekočine zapišemo v obliki:

lation (Boundary Element Method), in which the boundary values are imbedded within the formulation and whereby no additional expressions for boundary values of vorticity are necessary, which favorably influences the method’s stability [5] and [8].

The flow kinetics is given using the transport equation (4) of vorticity, which for newtonian fluids reads:

$$\frac{\partial \omega}{\partial t} + \left(\vec{\upsilon} \cdot \vec{\nabla}\right)\omega = \nu \Delta \omega \qquad (10).$$

Vrtinčna enačba za newtonske tekočine je numerično bolje raziskana kakor za nenewtonske, zato je smiselno računati nenewtonsko vrtinčno polje kot:

Since newtonian flows were analyzed much more frequently in the past than non-newtonian ones, the following decomposition is applied:

$$\omega_{NN} = \omega_N + \left(\omega_{NN} - \omega_N\right) \qquad (11),$$

kjer smo uvedli indeksa N in NN za newtonske in nenewtonske tekočine. Če označimo $\left(\omega_{NN} - \omega_N\right)$ z $\hat{\omega}$, lahko zgornjo enačbo zapišemo v obliki:

where the indices N and NN stand for newtonian, and non-newtonian flows, respectively. Denoting the difference between both vorticity components (i.e. $\left(\omega_{NN} - \omega_N\right)$ with $\hat{\omega}$ the following equations holds:

$$\omega_{NN} = \omega_N + \hat{\omega} \qquad (12),$$

tako da lahko zapišemo:

and further

$$\frac{D\hat{\omega}}{Dt} = \vec{\nabla} \times \vec{\nabla} \cdot \underline{\tau}_{NN} - \nu \nabla^2 \omega_N \qquad (13)$$

oziroma v numerično stabilnejši obliki:

or in numerically more stable form

$$\frac{D\hat{\omega}}{Dt} - \nu \nabla^2 \hat{\omega} = \vec{\nabla} \times \vec{\nabla} \cdot \underline{\tau} - \nu \nabla^2 \left(\omega_N + \hat{\omega}\right) \qquad (14).$$

Z vpeljavo brezdimenzijskih spremenljivk:

Using dimensionless variables

$$x = Lx^* \;,\; y = Ly^* \;,\; \upsilon_x = V\upsilon_x^* \;,\; \upsilon_y = V\upsilon_y^*$$
$$t = \frac{L}{V}t^* \;,\; \omega = \frac{V}{L}\omega^* \;,\; \psi = VL\psi^* \;,\; \tau = \rho V^2 \tau^* \qquad (15),$$

kjer sta L in V karakteristična dolžina in hitrost, lahko enačbi za newtonsko tekočino zapišemo v obliki, primerni za izračun z metodo končnih razlik:

with L and V characteristic length and velocity scale, respectively, the equations for newtonian flow can be written in the following form, suitable for finite difference calculations:

$$\begin{aligned}
&\omega_{Ni+1,j}^{*n+1}\left(-\frac{\psi^*_{Ni,j+1}-\psi^*_{Ni,j-1}}{4\Delta x^*\Delta y^*}+\frac{1}{Re\left(\Delta x^*\right)^2}\right)+\omega_{Ni-1,j}^{*n+1}\left(\frac{\psi^*_{Ni,j+1}-\psi^*_{Ni,j-1}}{4\Delta x^*\Delta y^*}+\frac{1}{Re\left(\Delta x^*\right)^2}\right)+\\
&\omega_{Ni,j+1}^{*n+1}\left(\frac{\psi^*_{Ni+1,j}-\psi^*_{Ni-1,j}}{4\Delta x^*\Delta y^*}+\frac{1}{Re\left(\Delta y^*\right)^2}\right)+\omega_{Ni,j+1}^{*n+1}\left(-\frac{\psi^*_{Ni+1,j}-\psi^*_{Ni-1,j}}{4\Delta x^*\Delta y^*}+\frac{1}{Re\left(\Delta y^*\right)^2}\right)+\\
&\omega_{Ni,j}^{*n+1}\left(-\frac{2}{Re\left(\Delta x^*\right)^2}-\frac{2}{Re\left(\Delta y^*\right)^2}+\frac{1}{\Delta t^*}\right)=\frac{\omega^{*n}_{Ni,j}}{\Delta t^*}
\end{aligned} \qquad (16),$$

$$\frac{\psi^*_{Ni+1,j}}{\left(\Delta x^*\right)^2}+\frac{\psi^*_{Ni-1,j}}{\left(\Delta x^*\right)^2}+\frac{\psi^*_{Ni,j+1}}{\left(\Delta y^*\right)^2}+\frac{\psi^*_{Ni,j-1}}{\left(\Delta y^*\right)^2}+\psi^*_{Ni,j}\left(-\frac{2}{\left(\Delta x^*\right)^2}-\frac{2}{\left(\Delta y^*\right)^2}\right)=-\omega^*_{Ni,j} \qquad (17),$$

kjer je Reynoldsovo število definirano kot:

therein, the *Re* (Reynolds number) is defined as

$$Re=\frac{VL}{\nu} \qquad (18).$$

$\hat{\omega}$ lahko izračunamo iz izrazov:

$\hat{\omega}$ can be calculated from the expressions:

$$\hat{\omega}^{*n+1}_{i+1,j}\left(-\frac{\psi^*_{i,j+1}-\psi^*_{i,j-1}}{4\Delta x^*\Delta y^*}+\frac{1}{Re\left(\Delta x^*\right)^2}\right)+\hat{\omega}^{*n+1}_{i-1,j}\left(+\frac{\psi^*_{i,j+1}-\psi^*_{i,j-1}}{4\Delta x^*\Delta y^*}+\frac{1}{Re\left(\Delta x^*\right)^2}\right)+$$
$$\hat{\omega}^{*n+1}_{i,j+1}\left(+\frac{\psi^*_{i+1,j}-\psi^*_{i-1,j}}{4\Delta x^*\Delta y^*}+\frac{1}{Re\left(\Delta y^*\right)^2}\right)+\hat{\omega}^{*n+1}_{i,j-1}\left(-\frac{\psi^*_{i+1,j}-\psi^*_{i-1,j}}{4\Delta x^*\Delta y^*}+\frac{1}{Re\left(\Delta y^*\right)^2}\right)+$$
$$\hat{\omega}^{*n+1}_{i,j}\left(-\frac{2}{Re\left(\Delta x^*\right)^2}-\frac{2}{Re\left(\Delta y^*\right)^2}+\frac{1}{\Delta t^*}\right)=\frac{\hat{\omega}^{*n}_{i,j}}{\Delta t^*}+\frac{\hat{\omega}^{*n}_{i+1,j}-2\hat{\omega}^{*n}_{i,j}+\hat{\omega}^{*n}_{i-1,j}}{Re\left(\Delta x^*\right)^2}+ \qquad (19),$$
$$\frac{\hat{\omega}^{*n}_{i,j+1}-2\hat{\omega}^{*n}_{i,j}+\hat{\omega}^{*n}_{i,j-1}}{Re\left(\Delta y^*\right)^2}+\frac{\hat{\omega}^{*n}_{Ni+1,j}-2\hat{\omega}^{*n}_{Ni,j}+\hat{\omega}^{*n}_{Ni-1,j}}{Re\left(\Delta x^*\right)^2}+\frac{\hat{\omega}^{*n}_{Ni,j+1}-2\hat{\omega}^{*n}_{Ni,j}+\hat{\omega}^{*n}_{Ni,j-1}}{Re\left(\Delta y^*\right)^2}+$$
$$\frac{\partial^2\tau^{*n}_{xx}}{\partial x^*\partial y^*}-\frac{\partial^2\tau^{*n}_{yy}}{\partial x^*\partial y^*}-\frac{\partial^2\tau^{*n}_{xy}}{\partial x^{*2}}+\frac{\partial^2\tau^{*n}_{xy}}{\partial y^{*2}}$$

$$\frac{\hat{\psi}^*_{i+1,j}}{\left(\Delta x^*\right)^2}+\frac{\hat{\psi}^*_{i-1,j}}{\left(\Delta x^*\right)^2}+\frac{\hat{\psi}^*_{i,j+1}}{\left(\Delta y^*\right)^2}+\frac{\hat{\psi}^*_{i,j-1}}{\left(\Delta y^*\right)^2}+\hat{\psi}^*_{i,j}\left(-\frac{2}{\left(\Delta x^*\right)^2}-\frac{2}{\left(\Delta y^*\right)^2}\right)=-\hat{\psi}^*_{i,j} \qquad (20).$$

Zadnji štirje členi $\left(\tau_{ij}\right)$ v enačbi (19) so podani s konstitutivnim modelom (zakonom tečenja).

Poznamo več preprostejših zakonov tečenja. Pri the zakonih je na splošno $\underline{\tau}$ podan z izrazom $\underline{\tau}=-\eta(\dot{\gamma})\underline{\dot{\gamma}}$, kjer lahko tenzor deformacijske hitrosti zapišemo kot:

The last four terms $\left(\tau_{ij}\right)$ in equation 19 are modeled using appropriate constitutive relationships for non-newtonian flows.

Several forms of constitutive relationships for non-newtonian flows are known. Therein, the shear is generally defined by $\underline{\tau}=-\eta(\dot{\gamma})\underline{\dot{\gamma}}$, while the deformation tensor can be written as:

$$\underline{\dot{\gamma}}=\vec{\nabla}\otimes\vec{\upsilon}+\left(\vec{\nabla}\otimes\vec{\upsilon}\right)^T \qquad (21)$$

in deformacijsko hitrost podamo z izrazom:

and rate of deformation presented as:

$$\dot{\gamma}=\sqrt{\frac{1}{2}\left(\underline{\dot{\gamma}}:\underline{\dot{\gamma}}\right)} \qquad (22),$$

kjer smo z operatorjem : označili sled tenzorja. Uporabili smo naslednje modele tečenja:

where : is the trace of a tensor. The following constitutive relationships are investigated:

Potenčni zakon:

Power-law fluid:

$$\eta(\dot{\gamma})=m\dot{\gamma}^{n-1} \qquad (23),$$

kjer sta m in n vrednosti, podani za vsako tekočino posebej. Če postavimo $n = 1$ in $m = \mu$, velja izraz za newtonsko tekočino. Izraz je pogosto uporabljan v industrijski praksi, kjer zgornje območje krivulje $\eta(\dot{\gamma})$ nima bistvenega pomena. Pri številnih problemih imamo velike vrednosti kotne deformacije $\dot{\gamma}$, kar naredi hitro rastoči del krivulje zanimiv. V literaturi [7] je odvisnost običajno podana v diagramu log η-log $\dot{\gamma}$. Če je tekočina temperaturno odvisna, morata biti koeficienta m in n podana v funkciji temperature.

where m and n are values determined for each individual fluid. If $n = 1$ and $m = \mu$ the newtonian fluid is obtained. The expression is used particularly in industrial practice where the upper $\eta(\dot{\gamma})$ curve does not have a particular meaning. On the other hand, several applications involve high shear rate $\dot{\gamma}$, thus making the rapidly increasing part of the curve interesting. In most texts [7] the curve is listed as log η-log $\dot{\gamma}$. If the flow is isothermal, both parameters must be known as a function of temperature.

Model Carreau:

V modelu se pojavljajo štiri spremenljivke, in sicer: časovna konstanta λ, limitni vrednosti viskoznosti ($\dot{\gamma}=0$ in $\dot{\gamma}\to\infty$) η_o in η_∞, ter parameter n, ki ima enak pomen kakor pri potenčnem zakonu. Mnoge polimere in taline lahko uspešno opišemo z uporabo tega modela.

Carreau model:

$$\eta(\dot{\gamma})=\eta_\infty+(\eta_o-\eta_\infty)\left[1+(\lambda\dot{\gamma})^2\right]^{\frac{n-1}{2}} \tag{24}$$

As variables in the model, there are, time constant λ, the zero shear rate variable viscosity ($\dot{\gamma}=0$ and $\dot{\gamma}\to\infty$) η_o, infinite shear rate variable viscosity η_∞, and parameter n with a similar meaning as in power-law fluids. Several polymer solutions and melts can be successfully described using this model.

Model Ellis:

Ellis model:

$$\eta(\tau)=\frac{\eta_o}{1+\left(\frac{\tau}{\tau_{1/2}}\right)^{\alpha-1}} \tag{25}$$

Model vsebuje tri parametre, limitno vrednost viskoznosti η_o, parameter α, in $\tau_{1/2}$, ki je vrednost, pri kateri je $\eta=\eta_O/2$.

The model uses three parameters: zero shear rate variable viscosity η_o, parameter α, and parameter $\tau_{1/2}$, which denotes value and $\eta=\eta_O/2$.

Model Oldroyd-B:

Poleg preprostejših poznamo tudi zahtevnejše modele, na primer model Oldroyd-B:

Oldroyd-B:

This model is more sophisticated than the ones presented above, i.e.:

$$\underline{\tau}+\lambda_1\overset{\nabla}{\underline{\tau}}=\eta_o\left(\underline{\dot{\gamma}}+\lambda_2\overset{\nabla}{\underline{\dot{\gamma}}}\right) \tag{26}$$

kjer sta λ_1 in λ_2 $(0\le\lambda_2\le\lambda_1)$ relaksacijski in zakasnitveni faktor. Zgornji konvektivni odvod je definiran z izrazom:

while λ_1 and λ_2 $(0\le\lambda_2\le\lambda_1)$ are relaxation, and delay parameters, respectively. The upper convective derivative is defined as:

$$\overset{\nabla}{\underline{\tau}}=\frac{\partial\tau}{\partial t}+\vec{v}\cdot\vec{\nabla}\underline{\tau}-\underline{\tau}\cdot\vec{\nabla}\vec{v}-\left(\vec{\nabla}\vec{v}\right)^T\cdot\underline{\tau} \tag{27}$$

kar lahko zapišemo v komponentni obliki za primer ravninskega kartezičnega koordinatnega sistema:

or in its component form for the 2D Cartesian coordinate system:

$$\begin{aligned}
&\tau_{xx}+\lambda_1\left\{\frac{\partial\tau_{xx}}{\partial t}+v_x\frac{\partial\tau_{xx}}{\partial x}+v_y\frac{\partial\tau_{xx}}{\partial y}-2\left(\tau_{xx}\frac{\partial v_x}{\partial x}+\tau_{yx}\frac{\partial v_x}{\partial y}\right)\right\}=\\
&2\eta\left[\frac{\partial v_x}{\partial x}+\lambda_2\left\{\frac{\partial}{\partial t}\left(\frac{\partial v_x}{\partial x}\right)+v_x\frac{\partial^2 v_x}{\partial x^2}+v_y\frac{\partial^2 v_x}{\partial y\partial x}-2\left(\frac{\partial v_x}{\partial x}\frac{\partial v_x}{\partial x}+\frac{1}{2}\frac{\partial v_x}{\partial y}\left(\frac{\partial v_y}{\partial x}+\frac{\partial v_x}{\partial y}\right)\right)\right\}\right]\\
&\tau_{xy}+\lambda_1\left\{\frac{\partial\tau_{xy}}{\partial t}+v_x\frac{\partial\tau_{xy}}{\partial x}+v_y\frac{\partial\tau_{xy}}{\partial y}-\left(\tau_{xx}\frac{\partial v_y}{\partial x}+\tau_{yx}\frac{\partial v_y}{\partial y}+\tau_{yx}\frac{\partial v_x}{\partial x}+\tau_{yy}\frac{\partial v_x}{\partial y}\right)\right\}=\\
&2\eta\left[\frac{1}{2}\left(\frac{\partial v_y}{\partial x}+\frac{\partial v_x}{\partial y}\right)+\lambda_2\left\{\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial v_y}{\partial x}+\frac{\partial v_x}{\partial y}\right)+\frac{1}{2}v_x\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_x}{\partial x\partial y}\right)+\right.\right.\\
&\left.\left.\frac{1}{2}v_y\left(\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_y}{\partial x\partial y}\right)-\left(\frac{3}{2}\left(\frac{\partial v_x}{\partial x}\frac{\partial v_y}{\partial x}+\frac{\partial v_x}{\partial y}\frac{\partial v_y}{\partial y}\right)+\frac{1}{2}\left(\frac{\partial v_x}{\partial x}\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\frac{\partial v_y}{\partial y}\right)\right)\right\}\right]\\
&\tau_{yy}+\lambda_1\left\{\frac{\partial\tau_{yy}}{\partial t}+v_x\frac{\partial\tau_{yy}}{\partial x}+v_y\frac{\partial\tau_{yy}}{\partial y}-2\left(\tau_{xy}\frac{\partial v_y}{\partial x}+\tau_{yy}\frac{\partial v_y}{\partial y}\right)\right\}=\\
&2\eta\left[\frac{\partial v_y}{\partial y}+\lambda_2\left\{\frac{\partial}{\partial t}\left(\frac{\partial v_y}{\partial y}\right)+v_x\frac{\partial^2 v_y}{\partial x\partial y}+v_y\frac{\partial^2 v_y}{\partial y^2}-2\left(\frac{1}{2}\frac{\partial v_y}{\partial x}\left(\frac{\partial v_y}{\partial x}+\frac{\partial v_x}{\partial y}\right)+\frac{\partial v_y}{\partial y}\frac{\partial v_y}{\partial y}\right)\right\}\right]
\end{aligned} \tag{28}$$

Hitrostno in tlačno polje lahko pozneje izračunamo iz enačb (1) in (5).

The velocity and pressure field can be calculated using equations (1) and (5).

2 PRIMER

V prvem delu prikaza numeričnega simuliranja je predstavljena konvergenca rezultatov v odvisnosti od gostote mreže. Rezultate smo primerjali z vrednostmi iz literature [2]. Kot primerjalne vrednosti smo uporabili najnižje negativne vrednosti vodoravne in navpične komponente hitrosti na osrednjih linijah. Odstopanje od primerjalnih vrednosti smo definirali z izrazoma:

2 RESULTS

First, the authors wish to present convergence ratio of results depending on mesh size. Reference [2] was used as a reference. The results were compared using the lowest (i.e. maximum negative) values for the horizontal and vertical velocity component on centerlines (i.e.horizontal and vertical centerline). The difference from the reference value was computed using the following definitions:

$$\Delta\upsilon_x = \frac{\left|\upsilon_{x_{ref}} - \upsilon_x\right|}{\left|\upsilon_{x_{ref}}\right|}\cdot 100\%$$

$$\Delta\upsilon_y = \frac{\left|\upsilon_{y_{ref}} - \upsilon_y\right|}{\left|\upsilon_{y_{ref}}\right|}\cdot 100\% \qquad (29).$$

Rezultati za *Re* = 400 za različne gostote mrež so zbrani v naslednji preglednici. Z mrežo s 129 × 129 vozlišči smo za različne vrednosti *Re* števila dobili naslednja odstopanja:

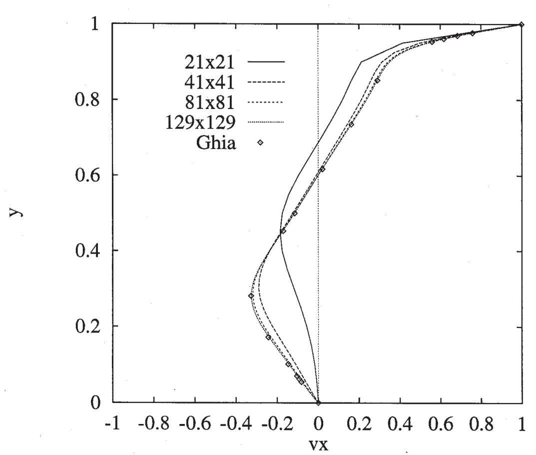
Preglednica 1. *Odstopanje najnižjih vrednosti hitrosti po srednjicah od referenčnih vrednosti za Re = 400*
Table 1. *The difference between present and reference values for Re = 400 and various mesh sizes*

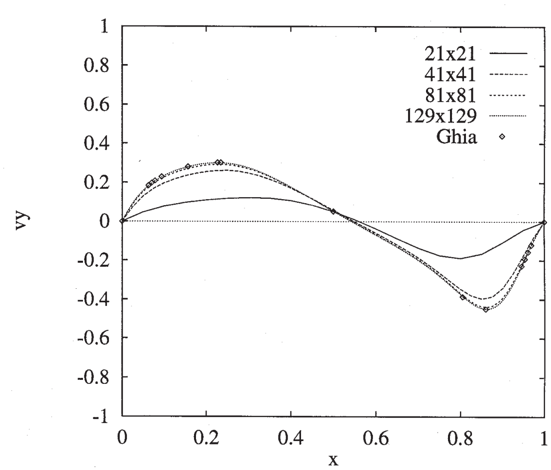
	$\Delta\upsilon_x$ %	$\Delta\upsilon_x$ %
21 × 21	69,23	65,45
41 × 41	12,94	12,90
81 × 81	2,92	2,59
129 × 129	0,57	0,23

Results for *Re* = 400 for different mesh sizes are presented in the following table. The results valid for 129 × 129 mesh size using various *Re* numbers are

Preglednica 2. *Odstopanje najnižjih vrednosti hitrosti po srednjicah od referenčnih vrednosti za različne vrednosti Re števila (mreža s 129 × 129 vozlišči)*
Table 2. *The difference between present and reference values for 129 × 129 mesh size and various Re numbers*

	$\Delta\upsilon_x$ %	$\Delta\upsilon_x$ %
100	0,63	0,31
400	0,57	0,23
1000	1,34	0,52
3200	5,55	3,82
5000	9,91	8,28





Sl. 1. *Profil vodoravne hitrosti υ_x skozi navpično središčnico (levo) in navpične hitrosti υ_y skozi vodoravno središčnico (desno) za različne gostote mrež (Re = 400)*
Fig. 1. *Profile of the horizontal component of velocity υ_x on vertical centerline (left) and vertical component of velocity υ_y on horizontal centerline (right) for various mesh sizes (Re = 400)*

Za manjše vrednosti *Re* števila (do Re = 1000) dobimo odstopanje v območju 1% z mrežo z 129×129 vozlišči, za višje vrednosti *Re* števila pa je treba za omenjeno natančnost mrežo še dodatno zgostiti.

Potek hitrostnih profilov v odvisnosti od gostote mreže za $Re = 400$ je prikazan na sliki 1.

Na slikah 2 do 5 so prikazana vrtinčna in tokovna polja za različne vrednosti *Re* števila, računana pri gostoti mreže 129×129 vozlišč. S slike 2 je razvidno, da že pri najnižji vrednosti *Re* števila ($Re = 100$) nastajata vrtinca v spodnjih dveh vogalih, ki z naraščanjem Re števila postajata intenzivnejša. Pri najvišjem računanem *Re* številu je opazno, da nastaja še dodatni vrtinec ob steni pod levim zgornjim vogalom.

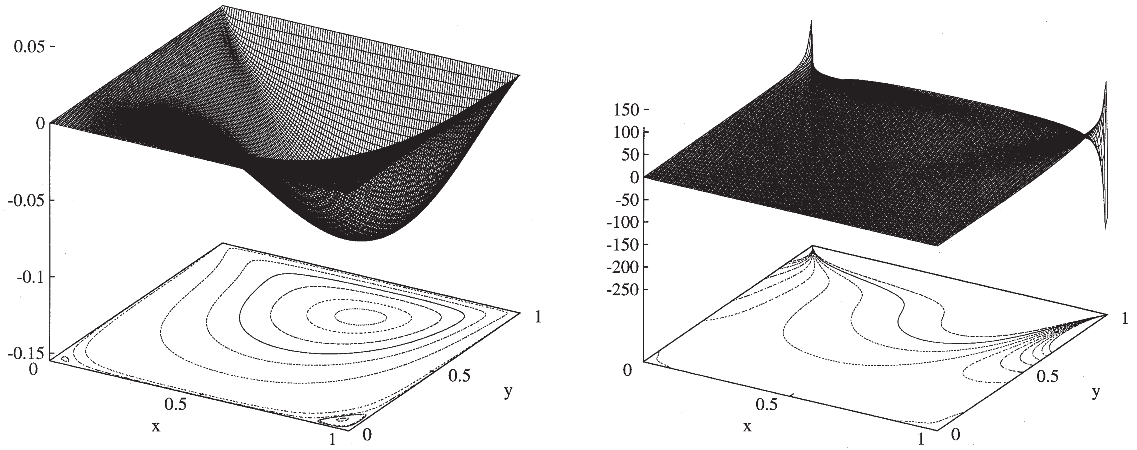
Na slikah 6 do 9 so prikazani profili za obe komponenti hitrosti po središčnici za različne vrednosti *Re* števila. Vidno je dobro ujemanje rezultatov za obe komponenti hitrosti za $Re \leq 1000$ in rahlo odstopanje profilov pri $Re = 5000$.

In the case of lower *Re* numbers (up to Re = 1000) the differences were below 1 % using 129×129 mesh size. For higher *Re* numbers finer mesh had to be used to retain similar levels of accuracy.

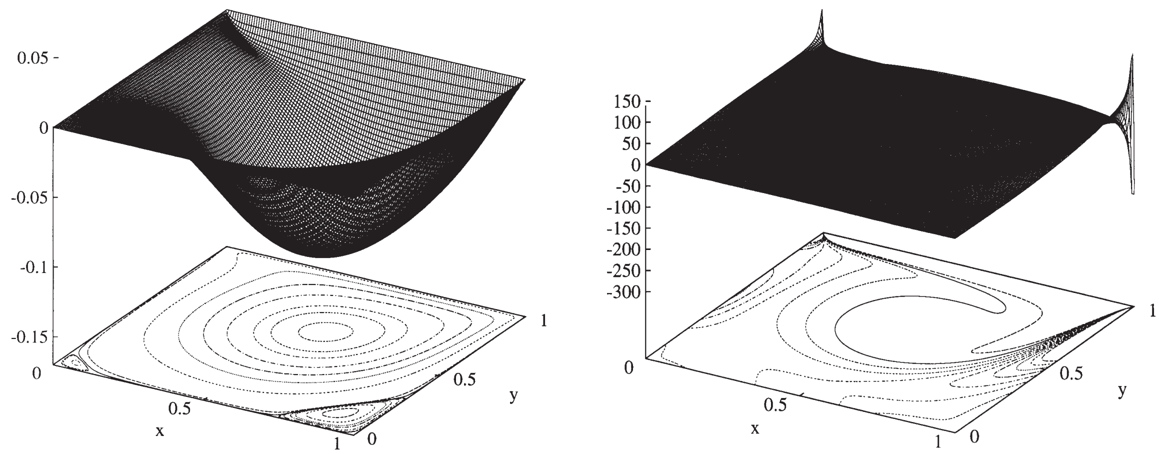
This is even better illustrated when velocity profiles on the vertical and horizontal centerline are plotted on same figure for various mesh sizes and $Re = 400$ (Fig. 1).

Figures 2 to 5 show the streamfunction and vorticity field for various *Re* numbers calculated on 129×129 mesh size. Figure 2 shows that vortices form at the lowest *Re* number. The size and intensity of these vortices increase with the increasing *Re* number. There is an additional vortex forming on the wall under the upper left corner for the highest *Re* number presented.

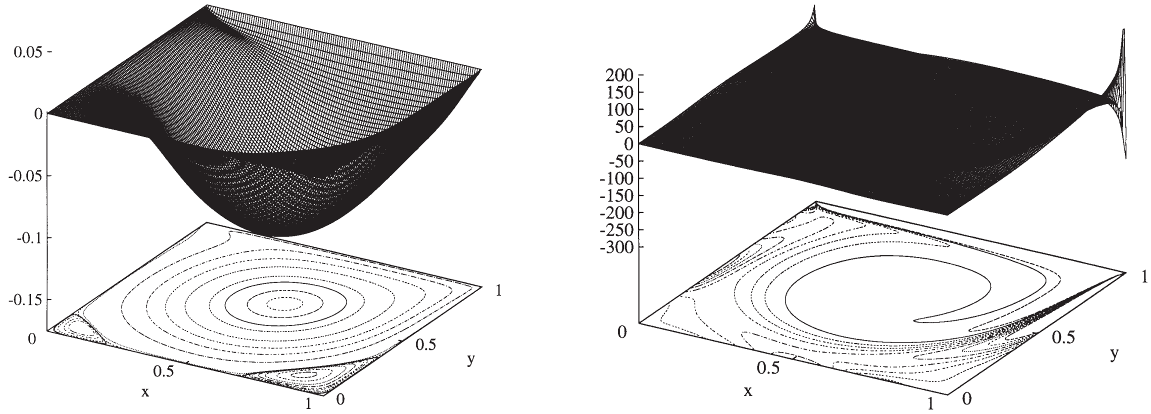
Figures 6 to 9 show velocity profiles for both velocity components on the centerline for various values of *Re* number. One notes good agreement between the presented and reference results for $Re \leq 1000$, and slight disagreement (attributable to too coarse mesh) for $Re = 5000$.



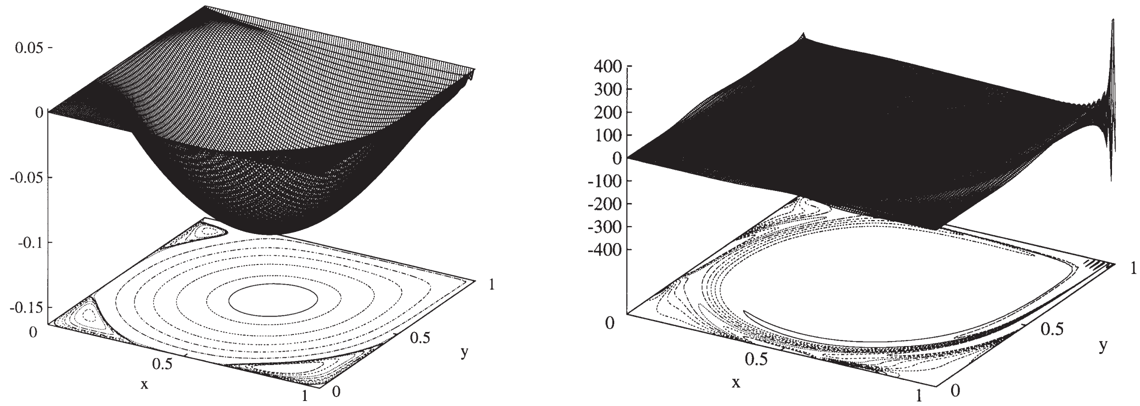
Sl. 2. *Tokovnice (levo) in vrtinčno polje (desno) v gnani kotanji za Re = 100*
Fig. 2. *Streamfunction (left) and vorticity field (right) in the driven cavity for Re = 100*



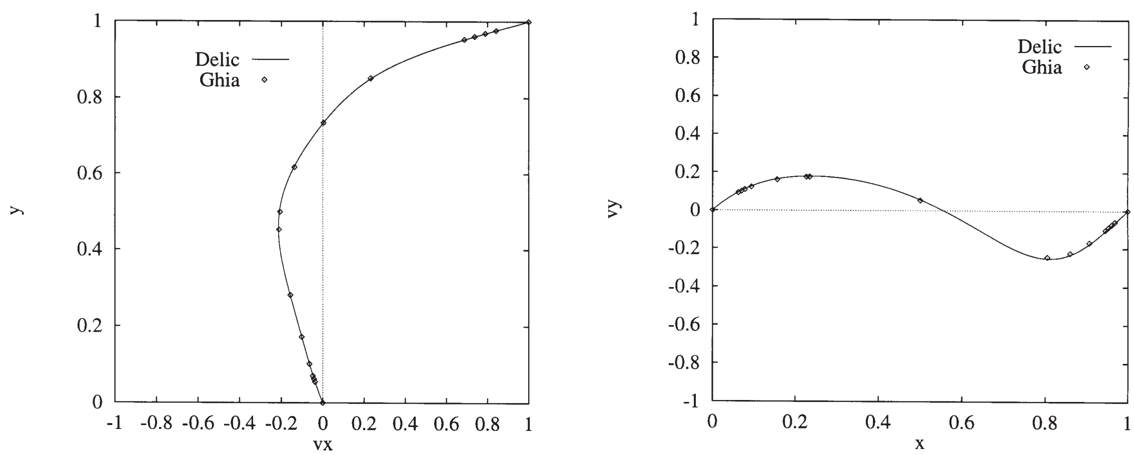
Sl. 3. *Tokovnice (levo) in vrtinčno polje (desno) v gnani kotanji za Re = 400*
Fig. 3. *Streamfunction (left) and vorticity field (right) in the driven cavity for Re = 400*



Sl. 4. *Tokovnice (levo) in vrtinčno polje (desno) v gnani kotanji za Re = 1000*
Fig. 4. *Streamfunction (left) and vorticity field (right) in the driven cavity for Re = 1000*



Sl. 5. *Tokovnice (levo) in vrtinčno polje (desno) v gnani kotanji za Re = 5000*
Fig. 5. *Streamfunction (left) and vorticity field (right) in the driven cavity for Re = 5000*



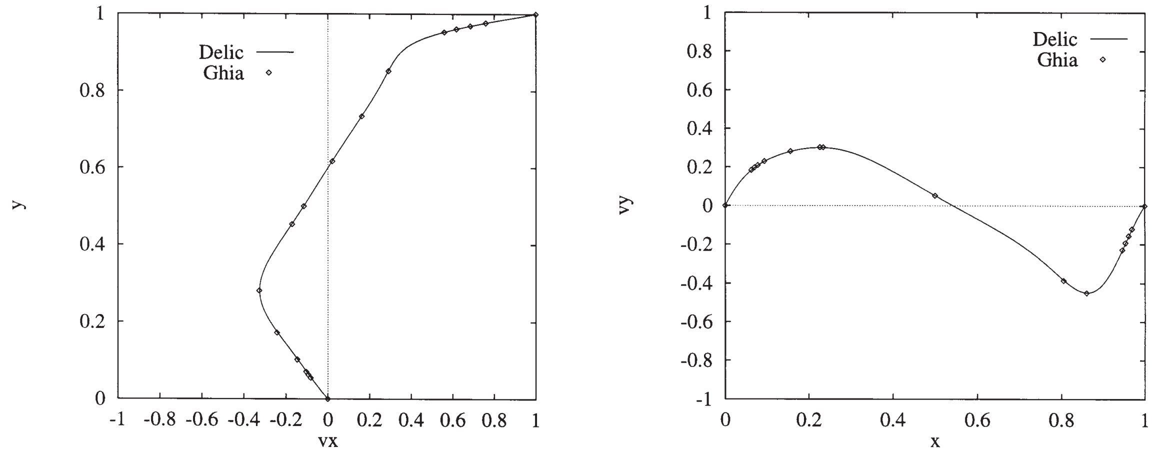
Sl. 6. *Profil vodoravne hitrosti* υ_x *skozi navpično središčnico (levo) in navpične hitrosti* υ_y *skozi vodoravno središčnico (desno) za Re = 100*
Fig. 6. *Profile of horizontal component of velocity* υ_x *on vertical centerline (left) and vertical component of velocity* υ_y *on horizontal centerline (right) for Re = 100*

3 SKLEP

V prispevku je prikazana formulacija ψ - ω metode končnih razlik za računanje tokovnih razmer v viskozni nestisljivi newtonski in nenewtonski tekočini. Formulacija je preverjena na primeru gnane kotanje za newtonske tekočine, kjer že pri majhnem *Re* številu pride do nastajanja dodatnih vrtincev, ki še povečajo zahtevnost numeričnega izračuna.

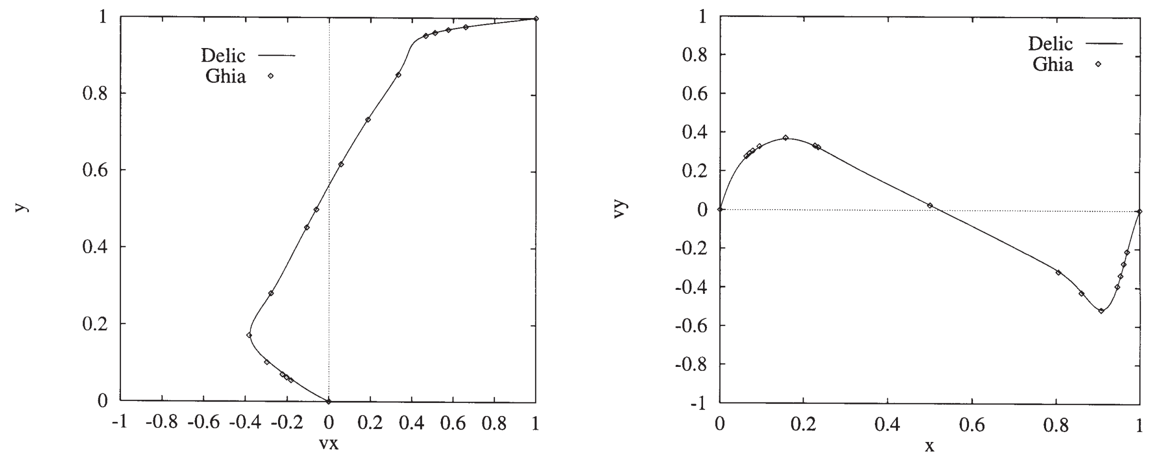
3 CONCLUSION

This paper deals with the ψ - ω formulation of a finite difference method to calculate incompressible viscous newtonian and non-newtonian fluid. The formulation is tested on driven cavity newtonian fluid flow. The results show vortex formation at low *Re* numbers, which increases the complexity of the nu-



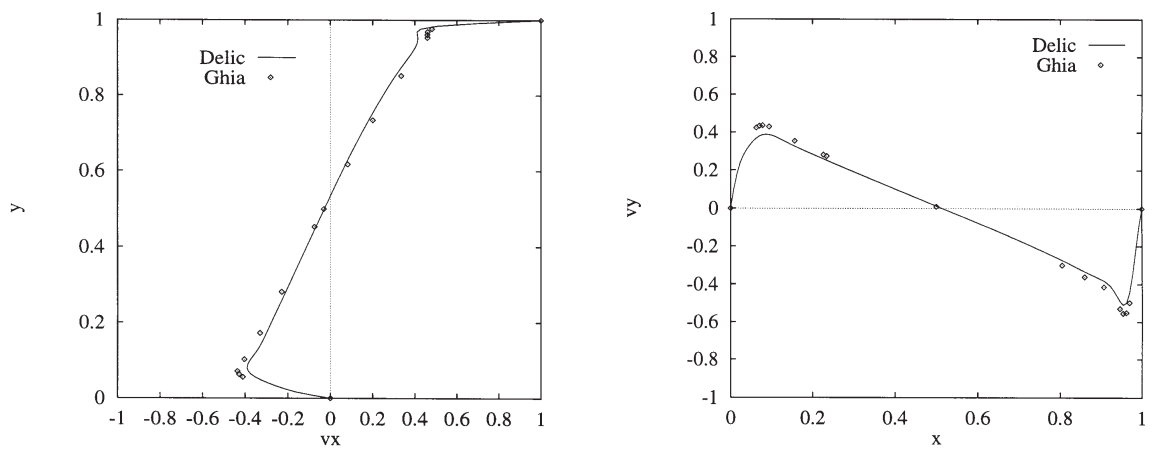
Sl. 7. *Profil vodoravne hitrosti* υ_x *skozi navpično središčnico (levo) in navpične hitrosti* υ_y *skozi vodoravno središčnico (desno) za Re = 400*

Fig. 7. *Profile of horizontal component of velocity* υ_x *on vertical centerline (left) and vertical component of velocity* υ_y *on horizontal centerline (right) for Re = 400*



Sl. 8. *Profil vodoravne hitrosti* υ_x *skozi navpično središčnico (levo) in navpične hitrosti* υ_y *skozi vodoravno središčnico (desno) za Re = 1000*

Fig. 8. *Profile of horizontal component of velocity* υ_x *on vertical centerline (left) and vertical component of velocity* υ_y *on horizontal centerline (right) for Re = 1000*



Sl. 9. *Profil vodoravne hitrosti* υ_x *skozi navpično središčnico (levo) in navpične hitrosti* υ_y *skozi vodoravno središčnico (desno) za Re = 5000*

Fig. 9. *Profile of horizontal component of velocity* υ_x *on vertical centerline (left) and vertical component of velocity* υ_y *on horizontal centerline (right) for Re = 5000*

Prikazan je vpliv natančnosti rezultatov v odvisnosti od gostote mreže. Prikazani rezultati, dobljeni z najgostejšo mrežo (129×129 vozlišč) za $Re \leq 1000$, odstopajo v območju 1% od referenčnih vrednosti, za višje vrednosti *Re* števila pa je za omenjeno natančnost mrežo treba še dodatno zgostiti.

merical calculations. The accuracy as a function of mesh size is shown. The results obtained with the finest (129×129) mesh for $Re \leq 1000$ differ by less than 1 % from the reference values, while for higher *Re* numbers the mesh should be refined.

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4 REFERENCES

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