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Investigation of Stress Concentration Factor for Finite-Width Orthotropic Rectangular Plates with a Circular Opening Using Three-Dimensional Finite Element Model

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In this paper, theoretical stress concentration factor for finite-width orthotropic plates with centered circular opening is investigated by use of three-dimensional finite element analysis. In the first part, the finite element (FE) model is validated by comparison analytic equations for infinite orthotropic and isotropic plates. With verified model, the stress concentration factor is calculated for orthotropic rectangular plates with centered circular opening subject to uniform tension. The effect of opening radius to width of plate ratio for various orthotropic plates with different orthotropy ratia is investigated. This study shows the accuracy of analytical method is dependent on orthotropy ratio and a/W ratio. In high opening radius to plate width ratio, the orthotropy ratio has a little effect on stress concentration factor.

Keywords: stress concentration factor, finite element methods, orthotropic plates

0 INTRODUCTION

Stress concentrations around cutouts have great practical importance because they are normally the cause of failure. For most materials, the failure strengths of the materials are strongly notch (or hole) sensitive. The net failure stress, taking into account the reduction in cross-section area, is typically much less than the ultimate tensile strength of the same materials without the notch or hole. Until now, the stress concentration factor for various isotropic structures is investigated. Heywood [1], Peterson [2] and Pilkey [3] have investigated various isotropic shape with wide range of holes. Howland [4] determined the solution of the problem of a long isotropic rectangular plate with a centered hole subject to a tension load. Peterson and Heywood introduced various equations for long length and finite width plate with different opening shapes.

Lekhnitskii [5] and Tan [6] introduced various formulations for stress concentration in orthotropic materials. Lekhnitskii derives equations for infinite plate with circular holes. For finite-width orthotropic plate Tan introduces various equations. He derives these equations by use of equilibrium relations, but he did not investigate the effect of orthotropy ratio. The orthotropy ratio is defined as the ratio of two elastic moduli in both main directions. The analysis presented here suggested, that Tan's equations are applicable only for a specific range of orthotropy ratio.

In this paper, the effect of orthotropy ratio on the stress concentration factor of rectangular finite-width plates with circular opening is investigated. These plates are under action of unidirectional tension loads. In the first part, the finite element model is validated for infinite orthotropic and isotropic rectangular plate with circular opening. With validated model, the effect of orthotropy ratio on stress concentration factor is investigated.

1 THEORETICAL STRESS CONCENTRATION FACTOR FOR FINITE WIDTH PLATE

As is well known the theoretical stress concentration factor (TSCF) for normal stress is defined according to [2] and [7]:

$$K_T = \frac{\sigma_{\max}}{\sigma_{\text{nom}}} \quad (1),$$

where the stress $\sigma_{\max}$ represents the maximum stress to be expected in the member under the actual loads and the normal stress σ_{nom} is reference normal

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stress. In a rectangular plate under action of unidirectional tension load S (Fig. 1), σ_{nom} is calculated based on the whole cross-section of the plate (gross area) as [2]:

$$\sigma_{nom} = \frac{S}{2Wh} \qquad (2),$$

where:
S - traction force, uniform on edges of plate,
W - plate half-width,
h - plate thickness.

In this way, defined reference stress is in fact the far away field stress. TSCF calculated from Equations (1) and (2) is denoted by K_{Tg} and is very useful when investigating wide plates with small openings, i.e. when approaching infinite domain problems.

Some values of the TSCF have been compiled by some major references Peterson, Pilkey, Tan and Lekhnitskii, and are reproduced in a number of mechanical design texts used worldwide.

For a finite width plate, the stress concentration factor can be calculated by use of finite-width correction (FWC) factor [6] and [8]. FWC factor is a scale factor defined as the stress concentration factor of an infinite to a finite-width plate. Thus, the stress concentration of finite-width plate can be calculated with specification of FWC factor and stress concentration of infinite plate with similar opening and material (orthotropic or isotropic).

FWC factor for isotropic material is independent of material properties thus can be determined accurately by use of curve fitting technique. Whereas in orthotropic material, this factor must be obtained with stress analysis by use

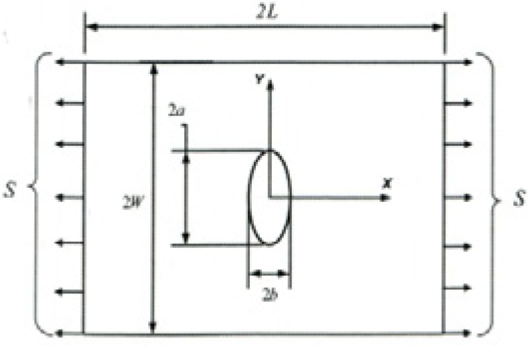

Fig. 1. *A finite-width plate containing a central elliptical opening*

of elasticity equations or mainly using finite element methods [6].

According to investigation of Tan on rectangular orthotropic plate with elliptical opening under action of unidirectional axial load (Fig. 1), the FWC factor is introduced in the form of following equation:

$$\frac{K_{Tg}^{\infty}}{K_{Tg}} = \frac{\lambda^2}{(1-\lambda)^2} + \frac{(1-2\lambda)}{(1-\lambda)^2}\sqrt{1+\left(\lambda^2-1\right)\left(\frac{a}{W}M\right)^2}$$
$$- \frac{\lambda^2}{(1-\lambda)}\left(\frac{a}{W}M\right)^2\left[1+(\lambda^2-1)\left(\frac{a}{W}M\right)^2\right]^{-1/2}$$
$$+\frac{\lambda^7}{2}\left(\frac{a}{W}M\right)^6\left(K_T^{\infty}-1-\frac{2}{\lambda}\right) \qquad (3),$$
$$\left(\begin{array}{l}\left[1+(\lambda^2-1)\left(\frac{a}{W}M\right)^2\right]^{-5/2} - \\ -\left(\frac{a}{W}M\right)^2\left[1+(\lambda^2-1)\left(\frac{a}{W}M\right)^2\right]^{-7/2}\end{array}\right)$$

where:

$$M^2 = \frac{\sqrt{1-8\left[\frac{3\left(1-\frac{a}{W}\right)}{2+\left(1-\frac{a}{w}\right)^3}-1\right]}-1}{2\left(\frac{a}{W}\right)^2} \qquad (4),$$

$\lambda = a/b$,
K_{Tg}^{∞} - stress concentration factor in infinite plate,
K_{Tg} - stress concentration factor in finite plate.

The magnification factor M, is only a function of a/W and is independent of material properties.

For a circular opening $\lambda = a/b = 1$ with using the L'Hospital's rule twice w.r.t λ [6], Equation (3) becomes:

$$\frac{K_{Tg}^{\infty}}{K_{Tg}} = \frac{3\left(1-\frac{a}{W}\right)}{2+\left(1-\frac{a}{W}\right)^3} + \frac{1}{2}\left(\frac{a}{W}M\right)^6\left(K_T^{\infty}-3\right)\left(1-\left(\frac{a}{W}M\right)^2\right) \qquad (5).$$

This equation can be used for orthotropic and isotropic finite-width plates with circular opening under action of unidirectional tension load for a/W up to 0.9 [6].

For calculation of stress concentration factor in a finite-width plate besides of FWC factor the stress concentration factor of infinite plate with similar opening must be calculated. For orthotropic plate, the stress concentration factor of finite-width plate is complicated with respect to isotropic plates. The Lekhnitskii [5] introduces stress concentration factor for orthotropic infinite plate, $K^{\infty}_{Tg,o}$ as below:

$$K^{\infty}_{Tg,o} = \frac{E_\theta}{E_{11}}\Big(\left[-\cos^2\varphi + (k+n)\sin^2\varphi\right]k\cos^2\theta + \\ +\left[(1+n)\cos^2\varphi - k\sin^2\varphi\right]\sin^2\theta - \\ -n(1+k+n)\sin\varphi\cos\varphi\sin\theta\cos\theta\Big) \quad (6),$$

where:

$$n = \sqrt{2\left(\sqrt{\frac{E_{11}}{E_{22}}} - \upsilon_{12} + \frac{E_{11}}{2G_{12}}\right)}$$

$$k = \sqrt{E_{11}/E_{22}} \quad (7).$$

$$\frac{1}{E_\theta} = \frac{\sin^4\theta}{E_{12}} + \left(\frac{1}{G_{12}} - \frac{2\upsilon_{12}}{E_{11}}\right)\sin^2\theta\cos^2\theta + \frac{\cos^4\theta}{E_{22}}$$

E_{11} and E_{22} are elasticity moduli in main directions, G_{12} is modulus of rigidity in shear plane, υ_{12} is Poisson ratio, θ is polar angle measured from X axis (main direction 1) and φ is the angle of acting force with respect to main direction 1 (Fig. 2).

For an infinite plate with circular opening under action of unidirectional tension, with direction of the applied force aligned with the main direction 1, φ is 0 and polar angle θ at points 'c' and 'd' on Figure 2 is 0 and 90 degree, respectively. The stress concentration factors for these two points are:

$$K^{\infty,1}_{Tg,o} = 1 + n \\ K^{\infty,2}_{Tg,o} = -k \quad (8),$$

where:

$K^{\infty,1}_{Tg,o}$ - stress concentration factor of point 'c' for orthotropic infinite plate,

$K^{\infty,2}_{Tg,o}$ - stress concentration factor of point 'd' for orthotropic infinite plate.

In isotropic material $E_{11} = E_{22}$ and $G = E/2(1+v)$, and according to Equations (6) and (7), the stress concentration factor for point 'c' and 'd' are 3 and -1, respectively. Substituting, $K^{\infty}_{Tg} = 3$ in the Equation (5), it is simplified to:

$$\frac{K^{\infty}_{Tg}}{K_{Tg}} = \frac{3(1-a/W)}{2+(1-a/W)^3} \quad (9).$$

This is familiar Heywood equation for isotropic infinite-plate with circular opening. Also, Howland determined the solution of the problem of a long rectangular plate with a central hole subject to tension in an open series form [4].

Another way to approaching the stress concentration factor and FWC factor is the application of the average normal stress across the net section ($2(W-a)h$) instead of the whole width of the plate (gross area). This stress concentration factor is known as net stress concentration factor, K_{Tn} and can be related to the previous stress concentration factor, K_{Tg} by:

$$K_{Tn} = K_{Tg}(1-a/W) \quad (10).$$

In the common engineering sense it is more convenient to give stress concentration factors using reference stress based on the net area rather than the gross area [9]. Thus in this study the net stress concentration factor is used.

1.1 Finite Element Model Validation

The stress concentration of an isotropic and orthotropic plate with circular opening under action of unidirectional tension is calculated by use of a finite element software MSC.Nastran [10].

Due to symmetry and using the appropriate displacement boundary conditions only one quarter of the plate was used as occupied by the shaded area indicated in Figure 3. The utilized restrictions

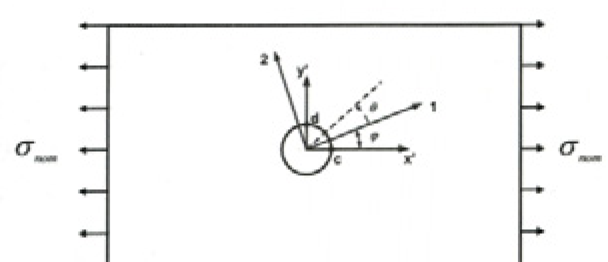

Fig. 2. *An infinite-plate containing a central circular opening*

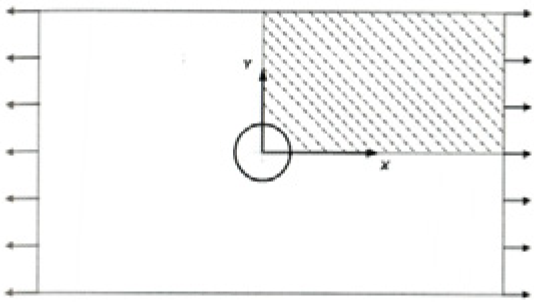

Fig. 3. *Finite element model region*

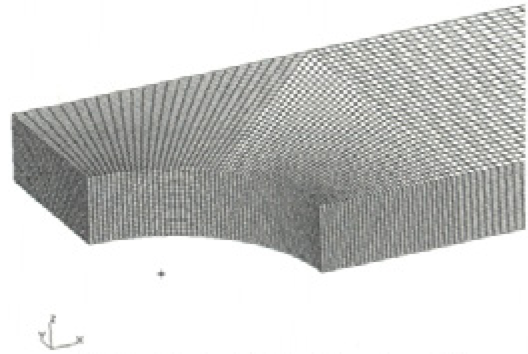

Fig. 4. *A typical finite element model*

are zero displacement in the X direction for all the mesh nodal points located on the line $X = 0$, and zero displacement in the Y direction for all mesh nodal points, located on the line $Y = 0$.

The finite element model is discretized with twenty-node solid brick elements (Fig. 4).

Various mesh densities for each of plates are used to achieve a good mesh density for improving accuracy of results. The mesh density is changed from coarser to finer. On the average and roughly the fine mesh have ten times more nodes and elements than the coarse mesh. Densifying is iteratively repeated until the difference between the consecutive results becomes smaller than 0.35%. In Figure 5 the final discrepancy of the TSCF for various opening ratia is shown. This figure reflects the absolute value of the difference (percent) in the magnitude of the TSCF due to the mesh refinement. Between 67171 to 70501 elements are used in the study.

Because the isotropic material properties are not different in main directions, this is used for improving the finite element model accuracy besides of orthotropic material.

The finite element model is created for various opening ratia, on plates with dimensions: half-length $L = 1000$ mm, half width $W = 100$ mm and thickness 10 mm for isotropic and orthotropic material. The opening radius to half-length of plate ratio a/W is changed from 0.1 to 0.8.

The isotropic plate has elastic modulus $E = 210$ GPa, Poisson ratio $\nu = 0.3$ and mass density $\rho = 7800$ kg/m^3.

The net stress concentration factor is calculated for isotropic plates by use of Equations (9) and (10) and compared with FE method results.

Table 1 shows the net stress concentration factors that are concluded by FE and analytical method for isotropic plates.

The error e in this table is defined as:

$$e = \left| \frac{TSCF_{Theory} - TSCF_{FEM}}{TSCF_{Theory}} \right| \cdot 100 \qquad (11),$$

where $TSCF_{Theory}$ and $TSCF_{FEM}$ are, respectively, the

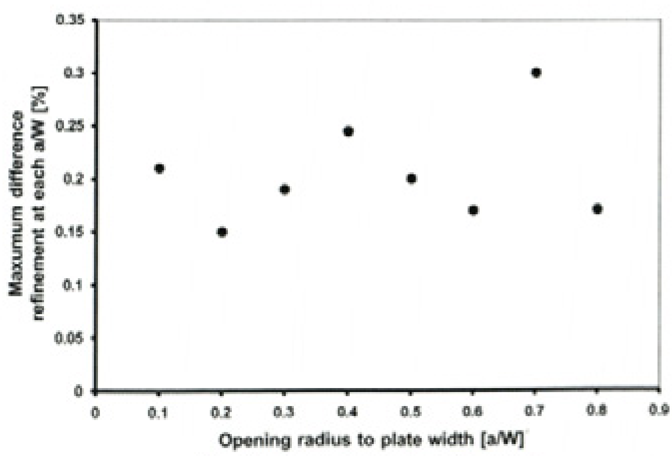

Fig. 5. *Convergence study results*

Table 1. *Net stress concentration factor for isotropic plate*

a/W	Net Stress Concentration Factor Heywood*	Howland**	FE	ERROR*** e %
0.1	2.72	2.72	2.73	0.37
0.2	2.61	2.62	2.59	0.61
0.3	2.34	2.36	2.35	0.31
0.4	2.22	2.24	2.23	0.35
0.5	2.12	2.16	2.11	0.41
0.6	2.06	2.09	2.05	0.29
0.7	2.03	2.05	2.04	0.24
0.8	2.01	2.03	2.02	0.40

* Eq. (9)
**Ref. [4]
*** Error (Eq. 11) of FE results with respect to Heywood results

stress concentration factors calculated with the analytic and FE method.

As it can be seen from Table 1 the agreement between the finite element method and theoretical solution for isotropic plate is excellent.

The distribution of net stress concentration factor in a vicinity of circular opening for isotropic plate with $a/W = 0.1$ is shown in Figure 6.

For orthotropic plate K_{Tn} is also calculated by use of FE and analytical method (Eqs. (5), (8) and (10)). The properties of the orthotropic material are:

Corresponding to a typical orthotropic material [11].

In Table 2, the calculated net stress concentration factor is compared. The error that is shown in this table is calculated by Equation (11).

Similar to isotropic plate, the FE method has a good accuracy for orthotropic materials.

The distribution of net stress concentration factor in a vicinity of circular opening for orthotropic plate with $a/W = 0.1$ is shown in Figure 7 .

$E_{11} = 44.7$ GPa, $E_{22} = E_{33} = 17.9$ GPa, $G_{12} = G_{13} = 8.96$ GPa,
$G_{23} = 3.45$ GPa, $\nu_{12} = 0.25$, $\nu_{13} = 0.25$,
$\nu_{23} = 0.34$, $\varphi = 0^{\circ}$,

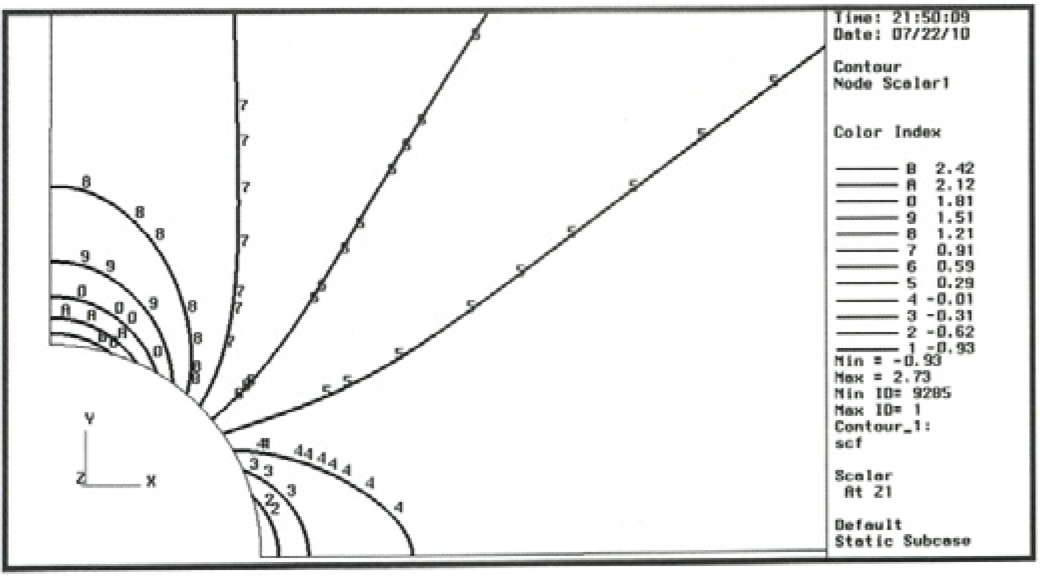

Fig. 6. *Net stress concentration factor distribution of isotropic plate with a/W = 0.1*

Table 2. *Net stress concentration factor for orthotropic plate*

a/W	Net Stress Concentration Factor Theoretical*	 FE	ERROR** e %
0.1	3.43	3.42	0.29
0.2	3.15	3.16	0.44
0.3	2.94	2.93	0.32
0.4	2.76	2.77	0.36
0.5	2.61	2.62	0.38
0.6	2.47	2.48	0.39
0.7	2.34	2.33	0.43
0.8	2.22	2.21	0.45

* Eqs. (5), (8), (10)
** Eq. (11)

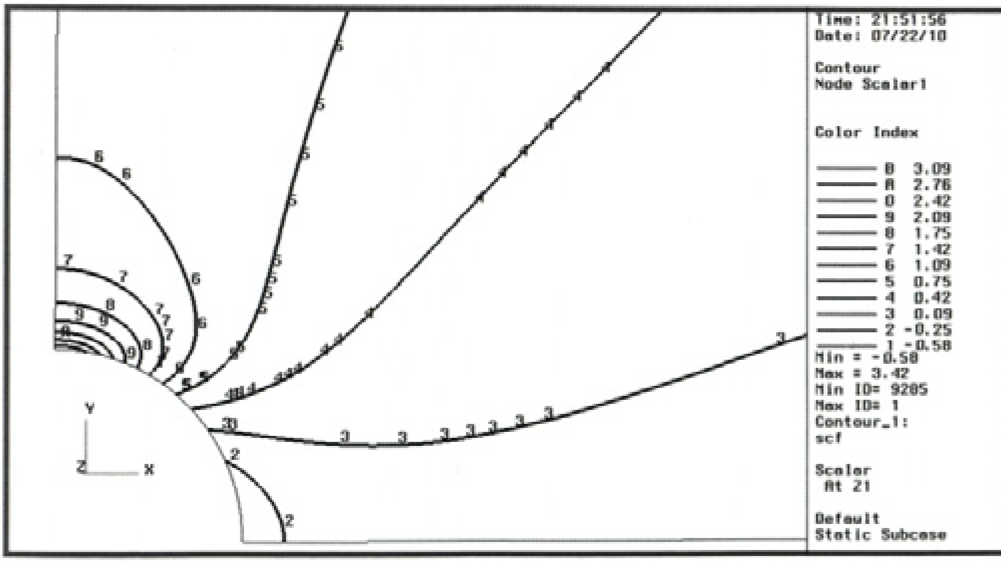

Fig. 7. *Net stress concentration factor distribution of orthotropic plate with a/W = 0.1*

2 ORTHOTROPIC RATIO EFFECT INVESTIGATION PROCEDURE

In the previous section, the FE method was validated for an orthotropic and isotropic finite-width plate with different circular opening radius to plate width ratio. In this section, the effect of orthotropy ratio on the stress concentration factor is investigated. The orthotropy ratio E_{11}/E_{22} is defined as the ratio of elastic modulii in main directions.

The procedure employed in the research consisted in determining stress concentration factor for a group of plates with a constant length to width ratio and different *a/W* and orthotropy ratio. The *a/W* ratio is changed from 0.1 to 0.8 and for each ratio the stress concentration factor is calculated for orthotropy ratio ranged from 2 to 50. In this research, the elastic modulus in width direction (E_{22}) is assumed to be constant and has a value of 17.9 GPa.

3 RESULTS AND DISCUSSION

The stress concentration factor for various *a/W* ratio and orthotropy ratio is calculated by FE method and compared with analytical results (5), (8) and (10). The absolute difference of FE method and analytical method is calculated by Equation 11 and shown in Figure 8.

Tan mentioned that his Equation (5) could be used for wide range of *a/W* ratio even up to 0.9. According to Figure 8 this equation's accuracy is not good for all range of *a/W* ratio. As shown in Figure 8 the analytical and FE results for $a/W \leq 0.4$

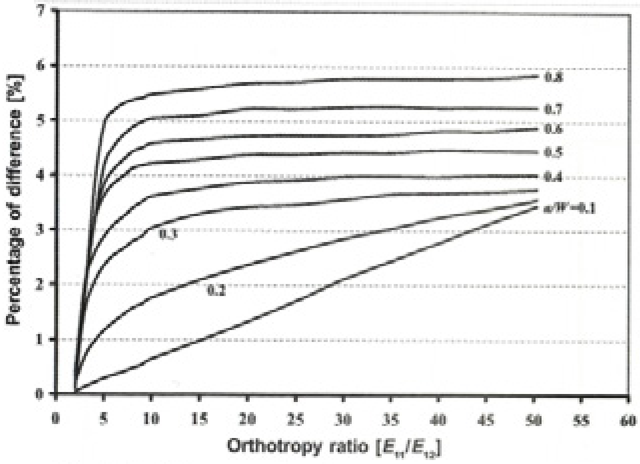

Fig. 8. *Difference of analytic and FE method results*

have a little difference for various orthotropy ratio (difference is less than 4%), but with increasing the a/W ratio this difference increases. In low orthotropy ratio the difference between analytical and FE results have a little difference, but this difference increases with increasing orthotropy ratio. Thus, the accuracy of Tan's Equation (5) decreases with increasing the a/W ratio and orthotropy ratio.

The variation of the net stress concentration factor with respects to a/W ratia for various orthotropy ratios is shown in Figure 9. According to this figure the higher orthotropy ratio have a higher stress concentration factor. In addition, the variation of SCF respect to a/W ratio in high orthotropy ratio is higher than at low orthotropy ratio. The important observations regarding the

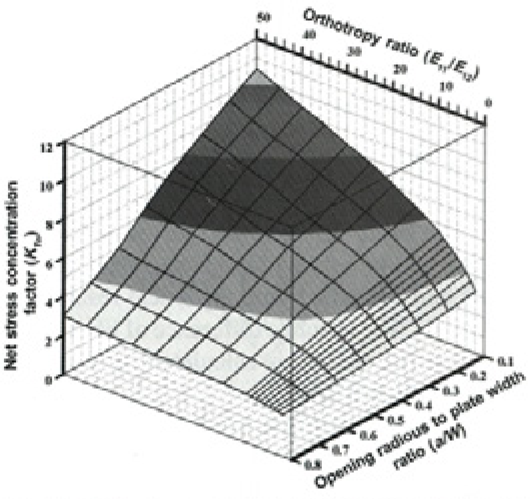

Fig. 9. *Net stress concentration factor respect to orthotropy ratio and a/W ratio*

graph of this figure are that all curves converge together with increasing the a/W ratio. Indeed, with increasing the a/W ratio, the effect of orthotropy ratio on net stress concentration factor is decreased.

According to Figure 9, the net stress concentration factor for low orthotropy ratio is near to each other and increasing this ratio the difference of a/W curves. The various a/W curves are converged to each other with low E_{11}/E_{22} ratio. In addition, with increasing the a/W ratio, the variation of orthotropy ratio has a little effect on the net stress concentration factor. This effect is also shown in Figure 9.

4 CONCLUSION

Formulation and comparative finite element results have been presented for a conforming finite element method, which may be used to model a plate with circular opening under action of tension load, and this comparison shows that appropriate finite element model has a good accuracy for stress concentration problems.

The stress concentration factor for low orthotropy ratio is near to each other and with increasing this ratio, the difference of a/W curves increases. The variation of net stress concentration factor, respect to orthotropy ratio increases with increasing the a/W ratio. In high a/W ratio the effect of orthotropic ratio is decreased, thus in high a/W ratio the elastic modulus changing in applied force direction has a little effect on the stress concentration factor. In other word: high orthotropy ratio and low opening ratio (a/W) increase net stress concentration factor.

The accuracy of analytical method is dependent on orthotropy ratio and *a/W* ratio and it is suitable for $a/W \leq 0.4$. Increasing the *a/W* ratio, this accuracy decreases further the influence of orthotropy ratio is small. At *a/W* ratio 0.8, discrepancy between analytical and numerical method is less than 6% , which is acceptable in engineering applications. Therefore, Tan's Equation (5) can be used instead of tedious and time consuming finite element analyses for all investigated ratia (up to 0.8).

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