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Theory and Application of Naturally Curved and Twisted Beams with Closed Thin-Walled Cross Sections

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A novel theory for analyzing naturally curved and twisted beams with closed thin-walled cross sections is presented based on the small displacement theory. By introducing the eigenwarping functions and expanding axial displacements or axial stress distribution in a series of eigenwarpings, the differential equation for determining generalized warping coordinate and the expression for eigenvalues can be derived from the principle of minimum potential energy. In the derivation procedure, the effects of the initial torsion and small initial curvature of the beams are accurately taken into account. The non-classical influences relevant to the beams are transverse shear and torsion-related warping deformations. Improved solutions can be obtained by adding a series expansion in terms of eigenwarpings to the uncorrected solution. The present theory is used to investigate the stresses and displacements of a cantilevered, rectangular box curved beam subjected to a uniformly distributed load. It is observed that the numerical results obtained agree well with the data from FEM.

Keywords: naturally curved and twisted beam, small initial curvature, small displacement theory, eigenwarping, generalized warping coordinate

0 INTRODUCTION

Static and dynamic analysis of naturally curved and twisted beams with closed thin-walled cross sections has many important applications in mechanical, civil and aeronautical engineering due to their outstanding engineering properties, such as streamlined modeling and favorable loaded characteristics. The structural behavior of the beams is no longer appropriately modeled with the classical beam theory ([1] to [3]), and a more advanced theory is much needed to overcome the demerits of the classical beam theory. Much research has been done in the theories for straight beams and curved beams ([4] to [14]), however, much less has been done for naturally curved and twisted beams. Bauchau and Bauchau et al. ([15] and [16]) provided a comprehensive treatment to the problem of warping using variational principles to model thin-walled straight beams made of anisotropic materials, however, their modes can not be used for naturally curved and twisted beams straightly. Based on small displacement theory, the main contribution of the present work is to derive a set of orthonormal eigenwarpings and equivalent constitutive equations that can be used for the analysis of naturally curved and twisted beams. In addition, the correction to transverse shear deformations is also included in the present formulations.

1 GEOMETRY AND CONSTITUTIVE RELATIONS OF THE BEAM

Let the locus of the cross-sectional centroid of the beam be a continuum curve l in space, the tangential, normal and bi-normal unit vectors of the curve are t, n and b, respectively. The Frenet-Serret formula, for a smooth curve, is:

$$\boldsymbol{t}' = k_1\boldsymbol{n} \quad \boldsymbol{n}' = -k_1\boldsymbol{t} + k_2\boldsymbol{b} \quad \boldsymbol{b}' = -k_2\boldsymbol{n}, \tag{1}$$

where ( )' means derivative with respect to s. s, k_1 and k_2 are arc coordinate, curvature and torsion of the curve, respectively.

We introduce ξ - and η - directions in coincidence with the principal axes through the centroid O_1, as shown in Fig. 1. The angle between the ξ - axis and normal n is represented by θ, which is generally a function of s. If the unit vectors of $O_1\xi$ and $O_1\eta$ are represented by i_ξ and i_η, then:

$$\boldsymbol{i}_\xi = \boldsymbol{n}\cos\theta + \boldsymbol{b}\sin\theta$$
$$\boldsymbol{i}_\eta = -\boldsymbol{n}\sin\theta + \boldsymbol{b}\cos\theta. \tag{2}$$

From Eq. (1) the following expressions are obtained:

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$$\boldsymbol{t}' = k_\eta \boldsymbol{i}_\xi - k_\xi \boldsymbol{i}_\eta$$
$$\boldsymbol{i}_\xi{}' = -k_\eta \boldsymbol{t} + k_s \boldsymbol{i}_\eta$$
$$\boldsymbol{i}_\eta{}' = k_\xi \boldsymbol{t} - k_s \boldsymbol{i}_\xi \tag{3}$$

which $k_\xi = k_1 \sin\theta,\ k_\eta = k_1 \cos\theta,\ k_s = k_2 + \theta'$.

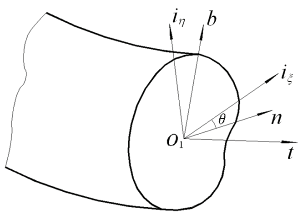

Fig. 1. *Geometry of the beam*

Fig. 2 depicts a cross section of the beam, where ζ is the curvilinear coordinate describing the contour of the section, denoted C. In the present work, the basic assumption is that the contour does not deform in its own plane, but is free to warp out of plane. This means that the degree of freedom of the deformation is fully represented by six rigid body modes $u_s(s)$, $u_\xi(s)$, $u_\eta(s)$, $\varphi_s(s)$, $\varphi_\xi(s)$ and $\varphi_\eta(s)$, respectively, the three translations and three rotations of the section, and any applied transverse load only induces membrane stresses in the structure, specifically an axial stress flow n, and a shear stress flow q. These two stress flows are acting in the plane of contour and are uniform across the thickness of the walls that form the cross-section. The constitutive relations are:

$$n = E \cdot t \cdot e$$
$$q = G \cdot t \cdot \gamma, \tag{4}$$

where t is the wall thickness, E is the modulus of elasticity and G is the shear modulus of the material, respectively, e and γ are the membrane axial strain and (engineering) shear strain, respectively.

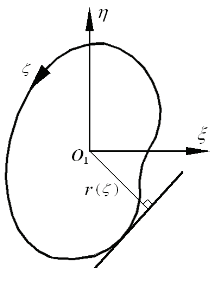

Fig. 2. *Closed cell thin-walled beam model*

2 INTERNAL FORCES, EQUILIBRIUM EQUATIONS AND KINEMATIC EQUATIONS

Simplifying stress vectors to the centroid O_1 on the cross section A, the principal vector $\boldsymbol{Q}$ and principal moment $\boldsymbol{M}$ can be obtained, of which components are respectively denoted by Q_s, Q_ξ, Q_η and M_s, M_ξ, M_η so:

$$\boldsymbol{Q} = Q_s \boldsymbol{t} + Q_\xi \boldsymbol{i}_\xi + Q_\eta \boldsymbol{i}_\eta,$$
$$\boldsymbol{M} = M_s \boldsymbol{t} + M_\xi \boldsymbol{i}_\xi + M_\eta \boldsymbol{i}_\eta,$$

where Q_s is axial force, Q_ξ and Q_η are shear forces, M_s is torque, M_ξ and M_η are bending moments, as shown in Fig. 3. The external forces and moments per unit length along the axis of the beam are indicated by $\boldsymbol{p}$ and $\boldsymbol{m}$ as:

$$\boldsymbol{p} = p_s \boldsymbol{t} + p_\xi \boldsymbol{i}_\xi + p_\eta \boldsymbol{i}_\eta,$$
$$\boldsymbol{m} = m_s \boldsymbol{t} + m_\xi \boldsymbol{i}_\xi + m_\eta \boldsymbol{i}_\eta.$$

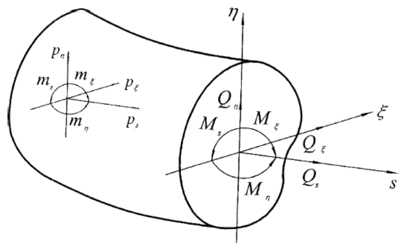

Fig. 3. *Stress resultants developed on a typical beam element*

The equilibrium equations are:

$$\frac{d}{ds}\{Q\} - [K]\cdot\{Q\} + \{p\} = \{0\},$$
$$\frac{d}{ds}\{M\} - [K]\cdot\{M\} - [H]\cdot\{Q\} + \{m\} = \{0\}, \tag{5}$$

where

$$\{Q\} = \begin{bmatrix} Q_s & Q_\xi & Q_\eta \end{bmatrix}^T,\ \{M\} = \begin{bmatrix} M_s & M_\xi & M_\eta \end{bmatrix}^T,$$
$$\{p\} = \begin{bmatrix} p_s & p_\xi & p_\eta \end{bmatrix}^T,\ \{m\} = \begin{bmatrix} m_s & m_\xi & m_\eta \end{bmatrix}^T,$$
$$[K] = \begin{bmatrix} 0 & k_\eta & -k_\xi \\ -k_\eta & 0 & k_s \\ k_\xi & -k_s & 0 \end{bmatrix},\ [H] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

The general solutions are [17]:

$$\{Q\} = [A] \cdot \left(\{Q_0\} - \int_0^s [A]^T \cdot \{p\} \, ds \right),$$

$$\{M\} = [A] \cdot \left\{ \{M_0\} + \int_0^s [A]^T \cdot \left([H] \cdot [A] \cdot \right.\right.$$
$$\left.\left. \left(\{Q_0\} + \{Q^*\} \right) - \{m\} \right) ds \right\}, \tag{6}$$

where $\{Q_0\}$ and $\{M_0\}$ are integration constants,

$$\{Q^*\} = -\int_0^s [A]^T \cdot \{p\} \, ds .$$

If the base vectors of special fixed right-handed rectangular coordinate system are i_x, i_y, i_z, then :

$$[A] = \begin{bmatrix} \boldsymbol{t} \cdot \boldsymbol{i}_x & \boldsymbol{t} \cdot \boldsymbol{i}_y & \boldsymbol{t} \cdot \boldsymbol{i}_z \\ \boldsymbol{i}_\xi \cdot \boldsymbol{i}_x & \boldsymbol{i}_\xi \cdot \boldsymbol{i}_y & \boldsymbol{i}_\xi \cdot \boldsymbol{i}_z \\ \boldsymbol{i}_\eta \cdot \boldsymbol{i}_x & \boldsymbol{i}_\eta \cdot \boldsymbol{i}_y & \boldsymbol{i}_\eta \cdot \boldsymbol{i}_z \end{bmatrix}. \tag{7}$$

The kinematic equations are:

$$\begin{aligned} \varepsilon_s &= u_s' - k_\eta u_\xi + k_\xi u_\eta , \\ \varepsilon_\xi &= u_\xi' + k_\eta u_s - k_s u_\eta - \varphi_\eta , \\ \varepsilon_\eta &= u_\eta' - k_\xi u_s + k_s u_\xi + \varphi_\xi , \\ \omega_s &= \varphi_s' - k_\eta \varphi_\xi + k_\xi \varphi_\eta , \\ \omega_\xi &= \varphi_\xi' + k_\eta \varphi_s - k_s \varphi_\eta , \\ \omega_\eta &= \varphi_\eta' - k_\xi \varphi_s + k_s \varphi_\xi , \end{aligned} \tag{8}$$

where ε_s, ε_ξ, ε_η, ω_s, ω_ξ, ω_η are respectively generalized strains corresponding to the internal forces Q_s , Q_ξ , Q_η , M_s , M_ξ, M_η and u_s, u_ξ, u_η, φ_s, φ_ξ, φ_η, are the displacement components corresponding to the loads p_s, p_ξ, p_η, m_s, m_ξ, m_η. The boundary conditions should be given by the following prescribed qualities:

$$Q_s \text{ or } u_s , Q_\xi \text{ or } u_\xi , Q_\eta \text{ or } u_\eta,$$
$$M_s \text{ or } \varphi_s, M_\xi \text{ or } \varphi_\xi, M_\eta \text{ or } \varphi_\eta. \tag{9}$$

Eqs. (8) can be rewritten as:

$$\frac{d}{ds}\{\varphi\} - [K]\{\varphi\} - \{\omega\} = \{0\},$$
$$\frac{d}{ds}\{u\} - [K] \cdot \{u\} - [H] \cdot \{\varphi\} - \{\varepsilon\} = \{0\}, \tag{10}$$

where:

$$\{\varphi\} = \begin{bmatrix} \varphi_s & \varphi_\xi & \varphi_\eta \end{bmatrix}^T, \ \{u\} = \begin{bmatrix} u_s & u_\xi & u_\eta \end{bmatrix}^T,$$

$$\{\varphi\} = \begin{bmatrix} \varphi_s & \varphi_\xi & \varphi_\eta \end{bmatrix}^T, \ \{u\} = \begin{bmatrix} u_s & u_\xi & u_\eta \end{bmatrix}^T,$$

$$\{\omega\} = \begin{bmatrix} \omega_s & \omega_\xi & \omega_\eta \end{bmatrix}^T, \{\varepsilon\} = \begin{bmatrix} \varepsilon_s & \varepsilon_\xi & \varepsilon_\eta \end{bmatrix}^T,$$

so the general solutions to the kinematic equations are [17]:

$$\{\varphi\} = [A] \cdot \left(\{\varphi_0\} + \{\varphi^*\} \right),$$

$$\{u\} = [A] \cdot \left\{ \{U_0\} + \int_0^s [A]^T \cdot \left(\{\varepsilon\} + [H] \cdot [A] \right.\right.$$
$$\left.\left. \cdot \left(\{\varphi_0\} + \{\varphi^*\} \right) ds \right\}, \tag{11}$$

in which $\{\varphi_0\}$ and $\{U_0\}$ are integration constants,

$$\{\varphi^*\} = \int_0^s [A]^T \cdot \{\omega\} \, ds .$$

3 STRUCTURAL ANALYSIS BY THE EIGENWARPING APPROACH

Eigenwarping theory is in fact to transform the solution of out-of-plane torsional warping of the cross-section into a problem for finding the eigenvaules and eigenvectors. The eigenvalue problem can be solved using a finite element technique where the eigenwarping function is discretized over the section of the beams. The solution to the problem will be improved by adding a series expansion in terms of eigenwarpings to the uncorrected solution. In addition, the correction to transverse shear deformations is also included in the present formulations.

Assuming that the deformations of the beam consist of stretching, bending and torsion, thus the displacement field can be written as follows:

$$\boldsymbol{u} = W\boldsymbol{t} + U\boldsymbol{i}_\xi + V\boldsymbol{i}_\eta , \tag{12}$$

in which:

$$W = u_s(s) + \eta\varphi_\xi(s) - \xi\varphi_\eta(s), \tag{12a}$$

$$U = u_\xi(s) - \eta\varphi_s(s), \qquad V = u_\eta(s) + \xi\varphi_s(s). \tag{12b}$$

The strain-displacement relations are [1]:

$$\begin{aligned} \sqrt{g} e_{11} &= \varepsilon_s + \eta\omega_\xi - \xi\omega_\eta , \\ 2\sqrt{g} e_{12} &= \varepsilon_\xi - \eta\omega_s , \\ 2\sqrt{g} e_{13} &= \varepsilon_\eta + \xi\omega_s , \end{aligned} \tag{13}$$

where ε_s, ε_ξ, ε_η, ω_s, ω_ξ, ω_η are the same as Eq. (8). For simplicity, the initial curvature k_1 is assumed small to assure that:

$$\sqrt{g} \approx 1.$$

The assumption is realistic for most practical applications, hence it does not seriously restrict the applicability of this model. In this development a set of orthonormal eigenwarpings that can be used for naturally curved and twisted beams is derived. An unloaded beam is now considered (i.e. $\{\boldsymbol{p}\}$, $\{\boldsymbol{m}\}$,=0) and a solution of the following form is assumed ([1] and [15]):

$$W(\zeta,s)=\varphi(\zeta)\alpha(s), \tag{14a}$$

$$\begin{aligned}\varepsilon_{\xi}(s)&=\bar{U}\alpha(s),\\ \varepsilon_{\eta}(s)&=\bar{V}\alpha(s),\\ \omega_{s}(s)&=\bar{\Xi}\alpha(s),\end{aligned} \tag{14b}$$

where $\varphi(\zeta)$ and $\alpha(s)$ are the eigenwarping modes of the cross-section and the generalized warping coordinates, respectively, and $\bar{U}$, $\bar{V}$ and $\bar{\Xi}$ are three unknown parameters. Substituting Eqs. (14) into the strain-displacement relations in [1], we obtain:

$$\begin{aligned}e_{11}&=\varphi\alpha'(s)+k_{s}\left[(\frac{\partial\varphi}{\partial\xi})\eta-(\frac{\partial\varphi}{\partial\eta})\xi\right]\alpha(s),\\ \gamma&=2e_{12}\frac{d\xi}{d\zeta}+2e_{13}\frac{d\eta}{d\zeta}\\ &=\bar{U}\alpha(s)\frac{d\xi}{d\zeta}-\eta\bar{\Xi}\alpha(s)\frac{d\xi}{d\zeta}\\ &\quad+\left[(\frac{\partial\varphi}{\partial\xi})+k_{\eta}\varphi\right]\alpha(s)\frac{d\xi}{d\zeta}\\ &\quad+\bar{V}\alpha(s)\frac{d\eta}{d\zeta}+\xi\bar{\Xi}\alpha(s)\frac{d\eta}{d\zeta}\\ &\quad+\left[(\frac{\partial\varphi}{\partial\eta})-k_{\xi}\varphi\right]\alpha(s)\frac{d\eta}{d\zeta}\\ &=(\frac{d\varphi}{d\zeta}-k_{\xi}\varphi\frac{d\eta}{d\zeta}+k_{\eta}\varphi\frac{d\xi}{d\zeta}\\ &\quad+\bar{U}\frac{d\xi}{d\zeta}+\bar{V}\frac{d\eta}{d\zeta}+r\bar{\Xi})\alpha(s),\end{aligned} \tag{15}$$

where the variables are separated, and r is the distance from the centroid O_1 to the tangent to the cross-sectional curve(see Fig. 2). The total strain energy in the beam is there:

$$\Pi=\frac{1}{2}\int_0^l\int_C(Ete_{11}^2+Gt\gamma^2)d\zeta\, ds,$$

where the product of the generalized warping coordinate and its derivative can be eliminated according to small displacement theory, and the differential equation defining generalized warping coordinates and associated eigenvalues can be derived by minimizing with respect to φ, $\bar{U}$, $\bar{V}$ and $\bar{\Xi}$. The derivation is similar to that in [15]. The associated eigenvalues μ_i^2 can also be written in the form of a Rayleigh quotient:

$$\mu_i^2=\frac{\int_C Gt\left\{\begin{array}{l}(\frac{d\varphi_i}{d\zeta}-k_{\xi}\varphi_i\frac{d\eta}{d\zeta}\\ +k_{\eta}\varphi_i\frac{d\xi}{d\zeta}+\bar{U}_i\frac{d\xi}{d\zeta}\\ +\bar{V}_i\frac{d\eta}{d\zeta}+\bar{\Xi}_i r)^2+\\ \frac{E}{G}k_s^2\left[\left(\frac{\partial\varphi_i}{\partial\xi}\right)\eta-\left(\frac{\partial\varphi_i}{\partial\eta}\right)\xi\right]^2\end{array}\right\}d\zeta}{\int_C Et\varphi_i^2 d\zeta}. \tag{16}$$

Unlike the case of straight beams, Eq. (16) contains the terms related to the initial curvature and torsion of the beam, and a set of orthonormality relationships for the present problem is:

$$\int_C Et\varphi_i\varphi_j d\zeta=\delta_{ij},\quad \int_C Gt\Gamma_i\Gamma_j d\zeta=\mu^2\delta_{ij}, \tag{17}$$

where:

$$\Gamma=\left(\left(\frac{d\varphi_i}{d\zeta}-k_{\xi}\varphi_i\frac{d\eta}{d\zeta}+k_{\eta}\varphi_i\frac{d\xi}{d\zeta}+\bar{U}_i\frac{d\xi}{d\zeta}+\bar{V}_i\frac{d\eta}{d\zeta}\right.\right.$$
$$\left.+\bar{\Xi}_i r)^2+\frac{E}{G}k_s^2\left[\left(\frac{\partial\varphi_i}{\partial\xi}\right)\eta-\left(\frac{\partial\varphi_i}{\partial\eta}\right)\xi\right]^2\right)^{\frac{1}{2}}.$$

4 IMPROVED BEAM THEORY AND EQUIVALENT CONSTITUTIVE EQUATIONS

Improved solutions will be obtained by using Eqs. (12a), (8) and (14) accordingly:

$$W=W_{or}+\sum_i\varphi_i\alpha_i, \tag{18a}$$

$$\begin{aligned}\varepsilon_{\xi}&=\varepsilon_{\xi or}+\sum_i\bar{U}_i\alpha_i(s),\\ \varepsilon_{\eta}&=\varepsilon_{\eta or}+\sum_i\bar{V}_i\alpha_i(s),\\ \omega_s&=\omega_{sor}+\sum_i\bar{\Xi}_i\alpha_i(s),\end{aligned} \tag{18b}$$

where W_{or} coincides with Eq. (12a), and $\varepsilon_{\xi\mathrm{or}}$, $\varepsilon_{\eta\mathrm{or}}$, ω_{sor} coincide with ε_{ξ}, ε_{η}, ω_{s} in Eq. (8). Thus, the final strain components e and γ are now:

$$e = e_{11} + \sum_i \varphi_i(\zeta)\alpha_i'(s) + k_s \sum_i \left[(\frac{\partial \varphi_i}{\partial \xi})\eta - (\frac{\partial \varphi_i}{\partial \eta})\xi \right] \alpha_i(s)$$
$$= \varepsilon_s + \eta\omega_\xi - \xi\omega_\eta + \sum_i \varphi_i(\zeta)\alpha_i'(s) + k_s \sum_i \left[(\frac{\partial \varphi_i}{\partial \xi})\eta - (\frac{\partial \varphi_i}{\partial \eta})\xi \right] \alpha_i(s),$$
$$\gamma = 2e_{12or}\frac{d\xi}{d\zeta} + 2e_{13or}\frac{d\eta}{d\zeta} + \sum_i \gamma_{co}$$
$$= \varepsilon_\xi \frac{d\xi}{d\zeta} + \varepsilon_\eta \frac{d\eta}{d\zeta} + r\omega_s + \sum_i (\frac{d\varphi_i}{d\zeta} - k_\xi \varphi_i \frac{d\eta}{d\zeta} + k_\eta \varphi_i \frac{d\xi}{d\zeta} + \bar{U}_i \frac{d\xi}{d\zeta} + \bar{V}_i \frac{d\eta}{d\zeta} + r\bar{\Xi}_\iota)\alpha_i(s), \tag{19}$$

where e_{11}, e_{12or}, e_{13or} coincides with Eq. (13). The strains corresponding to this displacement field can be used to evaluate the total potential energy Π:

$$\Pi = \frac{1}{2}\int_0^l \int_C (Ete^2 + Gt\gamma^2) d\zeta ds - \int_0^l (p_\xi u_\xi + p_\eta u_\eta + m_s \varphi_s) ds, \tag{20}$$

and taking into account the orthonormality relationships (17) it reads:

$$\Pi = \Pi_{or} + \sum_i \int_0^l \left[\frac{1}{2}(\alpha_i'^2 + \mu_i^2 \alpha_i^2) - d_i \alpha_i \right] ds, \tag{21}$$

where Π_{or} is the total potential energy for the original problem, and:

$$d_i = Q_\xi(\bar{U}_i + k_\eta \varphi_i) + Q_\eta(\bar{V}_i - k_\xi \varphi_i) + M_s \bar{\Xi}_i. \tag{22}$$

Eq. (21) shows the solutions to be decoupled from the corrective terms α_i and the different corrective terms decoupled from each other. Minimizing Π_{or} will render the equilibrium Eq. (5) described with the internal forces and minimizing the corrective terms with respect to α_i gives:

$$\alpha_i'' - \mu_i^2 \alpha_i = -d_i. \tag{23}$$

This differential equation is solved readily and yields the solution of the problem as a series expansion of eigenwarpings:

$$W = W_{or} + \sum_i \varphi_i \alpha_i, \qquad \varepsilon_\xi = \varepsilon_{\xi or} + \sum_i \bar{U}_i \alpha_i,$$
$$\varepsilon_\eta = \varepsilon_{\eta or} + \sum_i \bar{V}_i \alpha_i, \qquad \omega_s = \omega_{sor} + \sum_i \bar{\Xi}_i \alpha_i,$$
$$n = n_{or} + \sum_i Et\varphi_i \alpha_i' + k_s \sum_i Et \left[(\frac{\partial \varphi_i}{\partial \xi})\eta - (\frac{\partial \varphi_i}{\partial \eta})\xi \right] \alpha_i,$$
$$q = q_{or} + \sum_i Gt(\frac{d\varphi_i}{d\zeta} - k_\xi \varphi_i \frac{d\eta}{d\zeta} + k_\eta \varphi_i \frac{d\xi}{d\zeta} + \bar{U}_i \frac{d\xi}{d\zeta} + \bar{V}_i \frac{d\eta}{d\zeta} + r\bar{\Xi}_\iota)\alpha_i(s), \tag{24}$$

where n_{or} and q_{or} are the uncorrected solutions. When infinite series are used, Eq. (24) give the theoretical solution of the problem under the assumption of infinite in-plane rigidity of the section.

Introducing the internal forces defined by:

$$Q_s = \int_C n d\zeta, \qquad M_s = \int_C qr d\zeta,$$
$$Q_\xi = \int_C q\frac{d\xi}{d\zeta} d\zeta, \quad M_\xi = \int_C n\eta d\zeta,$$
$$Q_\eta = \int_C q\frac{d\eta}{d\zeta} d\zeta, \quad M_\eta = -\int_C n\xi d\zeta. \tag{25}$$

In view of Eqs. (4), (19) and (25), noting that:

$$\int_C Et\xi d\zeta = \int_C Et\eta d\zeta = \int_C Et\xi\eta d\zeta = 0,$$

we obtain the equivalent constitutive equations described with the generalized strains and generalized warping coordinates:

$$Q_s = S\varepsilon_s + \sum_i \int_C Et\varphi_i d\zeta \alpha_i' + k_s \sum_i \int_C Et \left[(\frac{\partial \varphi_i}{\partial \xi})\eta - (\frac{\partial \varphi_i}{\partial \eta})\xi \right] d\zeta \alpha_i,$$
$$Q_\xi = G_\xi A_{\xi\xi}\varepsilon_{\xi or} + G_\eta A_{\xi\eta}\varepsilon_{\eta or} + \int_C Gtr\frac{d\xi}{d\zeta} d\zeta \omega_{sor} + \sum_i \int_C Gt\Gamma_\iota \frac{d\xi}{d\zeta} d\zeta \alpha_i,$$
$$Q_\eta = G_\xi A_{\xi\eta}\varepsilon_{\xi or} + G_\eta A_{\eta\eta}\varepsilon_{\eta or} + \int_C Gtr\frac{d\eta}{d\zeta} d\zeta \omega_{sor} + \sum_i \int_C Gt\Gamma_\iota \frac{d\eta}{d\zeta} d\zeta \alpha_i,$$

$$M_s = I_P\omega_{sor} + \int_C Gtr\frac{d\xi}{d\zeta}d\zeta\varepsilon_{\xi or}$$
$$+\int_C Gtr\frac{d\eta}{d\zeta}d\zeta\varepsilon_{\eta or} + \sum_i \int_C Gt\Gamma_i r d\zeta\alpha_i,$$
$$M_\xi = I_{\xi\xi}\omega_\xi + \sum_i \int_C Et\eta\varphi_i d\zeta\alpha_i'$$
$$+k_s\sum_i \int_C Et\left[(\frac{\partial\varphi_i}{\partial\xi})\eta - (\frac{\partial\varphi_i}{\partial\eta})\xi\right]\eta d\zeta\alpha_i,$$
$$M_\eta = I_{\eta\eta}\omega_\eta - \sum_i \int_C Et\xi\varphi_i d\zeta\alpha_i'$$
$$-k_s\sum_i \int_C Et\left[(\frac{\partial\varphi_i}{\partial\xi})\eta - (\frac{\partial\varphi_i}{\partial\eta})\xi\right]\xi d\zeta\alpha_i, \quad (26)$$

where G_ξ and G_η are the shear coefficients in ξ- and η- directions for thin-walled beams [18], $S = \int_C Etd\zeta$ is the axial stiffness, $I_{\xi\xi} = \int_C Et\eta^2 d\zeta$ is the bending stiffness (similar definition for $I_{\eta\eta}$),

$A_{\xi\xi} = \int_C Gt(\frac{d\xi}{d\zeta})^2 d\zeta$ is the shear stiffness (similar definitions for $A_{\eta\eta}$ and $A_{\xi\eta}$), and $I_P = \int_C Gtr^2 d\zeta$ is the torsional stiffness. It is observed that the internal forces of the beam depend on not only the generalized strains but also the eigenwarping functions, generalized warping coordinates and their derivatives.

5 EXAMPLE - A CURVED BEAM UNDER A UNIFORMLY DISTRIBUTED LOAD

Some numerical results are given to demonstrate the theoretical formulations derived in previous sections, which will be directly applied to compute the stresses and displacements of a curved, thin-walled rectangular beam (see Fig. 4). In this case θ, k_s and k_ξ in Eq. (3) are zero and k_η is $1/R$. Fix the origin of the rectangular coordinate system at the end of the beam (s = 0), the axis of the beam being on the plane *Oxy*. The load acting is

$$\{m\} = [0 \quad 0 \quad 0]^T, \quad \{p\} = [0 \quad 0 \quad p_\eta]^T.$$

If the axis of the beam is a circle with radius α, one has

$$\beta = \frac{s}{a}, \qquad k_\eta = k_1 = \frac{1}{a},$$
$$x = a\sin\beta, \qquad y = a(1-\cos\beta).$$

Using Eqs. (6) and (11), we have:

$$M_s = M_{0s}\cos\beta + M_{0\xi}\sin\beta$$
$$+Q_{0\eta}a(1-\cos\beta) + p_\eta a^2(\sin\beta - \beta),$$
$$M_\xi = -M_{0s}\sin\beta + M_{0\xi}\cos\beta$$
$$+Q_{0\eta}a\sin\beta - p_\eta a^2(1-\cos\beta),$$
$$Q_\eta = Q_{0\eta} - p_\eta s,$$
$$\varphi_s = \varphi_{0s}\cos\beta + \varphi_{0\xi}\sin\beta$$
$$+a\cos\beta\int_0^\beta(\omega_s\cos\beta - \omega_\xi\sin\beta)d\beta$$
$$+a\sin\beta\int_0^\beta(\omega_s\sin\beta + \omega_\xi\cos\beta)d\beta,$$
$$\varphi_\xi = -\varphi_{0s}\sin\beta + \varphi_{0\xi}\cos\beta$$
$$-a\sin\beta\int_0^\beta(\omega_s\cos\beta - \omega_\xi\sin\beta)d\beta$$
$$+a\cos\beta\int_0^\beta(\omega_s\sin\beta + \omega_\xi\cos\beta)d\beta,$$
$$u_\eta = U_{0\eta} + \varphi_{0s}y - \varphi_{0\xi}x + a\int_0^\beta \varepsilon_\eta d\beta$$
$$+a\int_0^\beta\Big[a\sin\beta\int_0^\beta(\omega_s\cos\beta - \omega_\xi\sin\beta)d\beta$$
$$-a\cos\beta\int_0^\beta(\omega_s\sin\beta + \omega_\xi\cos\beta)d\beta\Big]d\beta, \quad (27)$$

where M_{0s}, $M_{0\xi}$, $Q_{0\eta}$ are the values of M_s, M_ξ, Q_η, at the end s = 0, and φ_{0s}, $\varphi_{0\xi}$, $U_{0\eta}$ are the values of φ_s, φ_ξ, u_η at s = 0, ω_s, ω_ξ, and ε_η are described with M_s, M_ξ, Q_η, α_i and α_i' by Eqs. (26).

The curved beam with radius α = 400 mm is assumed to be fixed at one end (s = 0) and free at the other (s = l). Fig. 4b illustrates the cross-section at the free end of the beam. The following properties of this material are:

$E = 2.106\cdot10^5$ MPa, $G = 81.6\cdot10^5$ MPa.

The first step is to calculate the eigenwarpings φ_i and the associated eigenvalues μ_i. The eigenvalue problem (16) can be solved using a finite element technique. For this example, 43 nodal points are used to model the section and 24 eigenwarpings are extracted.

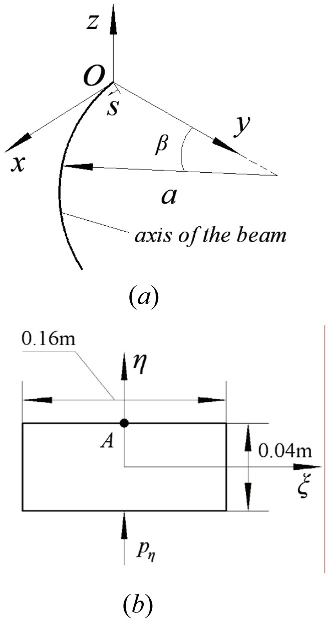

Fig. 4. *Axis and cross-section at the free end of a plane curved beam*

The bending and twisting behavior of the beam subjected to a uniformly distributed load p_η in the η direction will be described. In this case the differential Eq. (23) to be solved for improved solution is:

$$\alpha_i'' - \mu_i^2 \alpha_i = -p_\eta a \bar{V}_i (\frac{\pi}{2} - \beta) - p_\eta a^2 \bar{\Xi}_i (\frac{\pi}{2} - \beta - \cos\beta). \tag{28}$$

The solution to Eq. (28) must be:

$$\alpha_i = C_1 ch\mu_i s + C_2 sh\mu_i s + \alpha_i^*, \tag{29}$$

in which α_i^* is a particular solution to Eq. (28). The complete solution of Eq. (28) becomes:

$$\alpha_i = C_1 e^{\mu_i s} + C_2 e^{-\mu_i s} + \frac{1}{\mu_i^2} p_\eta a \bar{V}_i (\frac{\pi}{2} - \beta) + \frac{1}{\mu_i^2} p_\eta a^2 \bar{\Xi}_i (\frac{\pi}{2} - \beta) - \frac{1}{(\frac{1}{a^2} + \mu_i^2)} p_\eta a^2 \bar{\Xi}_i \cos\beta. \tag{30}$$

The boundary conditions are:

$$s = 0(\beta = 0), \quad U_{0s} = U_{0\xi} = U_{0\eta} = 0,$$
$$\varphi_{0s} = \varphi_{0\xi} = \varphi_{0\eta} = 0, \quad \alpha_i = 0,$$
$$s = l(\beta = \beta_l), \quad M_s = M_\xi = Q_\eta = 0,$$
$$\alpha_i' = 0,$$

where l = $\pi a/2$, the integration constants determined by the aforementioned conditions are:

$$M_{0s} = p_\eta a^2 (\frac{\pi}{2} - 1),$$
$$M_{0\xi} = -p_\eta a^2,$$
$$Q_{0\eta} = \frac{\pi}{2} p_\eta a,$$

$$C_1 = \left[-\frac{1}{2} \frac{a^3 \mu_i^3 \pi + \mu_i \pi a - 2e^{\frac{1}{2}\mu_i \pi a} \mu_i^2 a^2 - 2e^{\frac{1}{2}\mu_i \pi a}}{\mu_i^3 \left(1 + e^{\mu_i \pi a}\right)\left(1 + \mu_i^2 a^2\right)} \bar{V}_i - \frac{1}{2} \frac{a^4 \mu_i^3 \pi + a^2 \mu_i \pi - 2a^4 \mu_i^3 - 2e^{\frac{1}{2}\mu_i \pi a} a}{\mu_i^3 \left(1 + e^{\mu_i \pi a}\right)\left(1 + \mu_i^2 a^2\right)} \bar{\Xi}_i \right] p_\eta,$$

$$C_2 = \left[-\frac{1}{2} e^{\frac{1}{2}\mu_i \pi a} \times \frac{\left(2 + e^{\frac{1}{2}\mu_i \pi a} a^3 \mu_i^3 \pi + e^{\frac{1}{2}\mu_i \pi a} a \mu_i \pi + 2\mu_i^2 a^2\right)}{\mu_i^3 \left(1 + e^{\mu_i \pi a}\right)\left(1 + \mu_i^2 a^2\right)} \bar{V}_i - \frac{1}{2} e^{\frac{1}{2}\mu_i \pi a} \cdot \frac{\left(2a + e^{\frac{1}{2}\mu_i \pi a} a^4 \mu_i^3 \pi + e^{\frac{1}{2}\mu_i \pi a} a^2 \mu_i \pi - 2e^{\frac{1}{2}\mu_i \pi a} a^4 \mu_i^3\right)}{\mu_i^3 \left(1 + e^{\mu_i \pi a}\right)\left(1 + \mu_i^2 a^2\right)} \bar{\Xi}_i \right] p_\eta. \tag{31}$$

So far, the solutions to this problem have been obtained.

The determination of stresses and displacements is a significant problem in the static analysis of thin-walled beams. Let V_A present the displacement in the η - direction of point A on the cross section ($\beta = \pi/2$) of the beam due to the load p_η shown in Fig. 4b, and φ_s presents the tip twist angle of the beam. Theoretical results for V_A and φ_s are obtained and compared with a 2-D finite element analysis (referred as the FEM results), according to the ANSYS program. To analyze the beam shown in Fig. 4 by the finite element method, we partition it into 3600 shell elements (SHELL 92), and the total number of nodal points is 10880. These cases for p_η (0 → 1000 N/m) are shown in Fig. 5 a and b. It is evident that the theoretical results are close to the data from FEM.

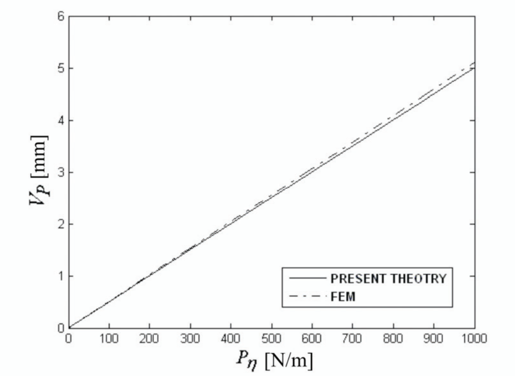

Fig. 5a. *Vertical displacement V_A of point A at the free end of the beam under a uniformly distributed load p_η (0 → 1000 N/m)*

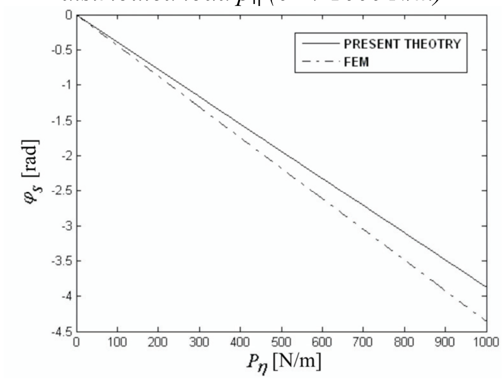

Fig. 5b. *Tip twist angle φ_s of the beam under a uniformly distributed load p_η (0 → 1000 N/m)*

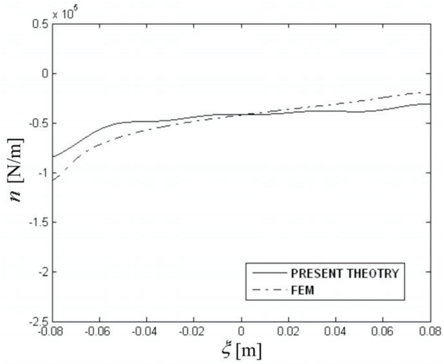

Fig.6a. *Distribution of the axial stress flow n in the upper face at the root of the beam under a uniformly distributed load p_η = 1000 N/m*

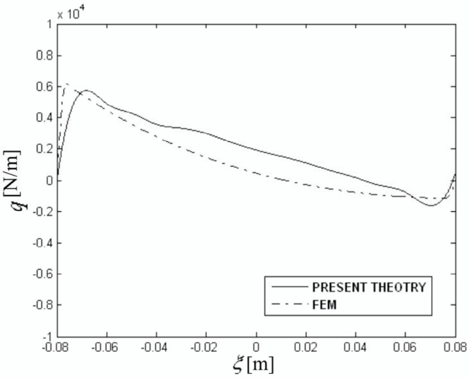

Fig. 6b. *Distribution of the shear stress flow q in the upper face at cross section (β = 45°) of the beam under a uniformly distributed load p_η = 1000 N/m*

It is also interesting to compute the stress distributions of the beam. Fig. 6 shows the distributions of the axial stress flow *n* in the upper face at the root (β = 0) and the shear stress flow *q* in the upper face at cross section (β = 45°) of the beam under a uniformly distributed load p_η = 1000N/m.

6 CONCLUSIONS

An analysis method for determining the stresses and displacements of naturally curved and twisted beams with closed thin-walled cross sections is developed based on small displacement theory. The effects of torsion-related warping, transverse shear deformations and extension-shearing coupling are included in the proposed model. The above results clearly indicate that the key factor to improving the stress and displacement predictive capability of a theory is to account for these non-classical effects correctly and find the eigenwarping modes of the cross-sections. The eigenwarpings depend on not only the material properties and geometry of the sections, but also the initial curvature and torsion of the beams. The theory suggested in this paper is not limited to thin-walled box beams. In the case of solid cross sections, the concept of eigenwarpings can be extended as long as the basic assumption remains valid, i. e., as long as the section can be assumed infinitely rigid in its own plane.

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