Theory and Application of Naturally Curved and Twisted Beams with Closed Thin-Walled Cross Sections

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A novel theory for analyzing naturally curved and twisted beams with closed thin-walled cross sections is presented based on the small displacement theory. By introducing the eigenwarping functions and expanding axial displacements or axial stress distribution in a series of eigenwarpings, the differential equation for determining generalized warping coordinate and the expression for eigenvalues can be derived from the principle of minimum potential energy. In the derivation procedure, the effects of the initial torsion and small initial curvature of the beams are accurately taken into account. The non-classical influences relevant to the beams are transverse shear and torsion-related warping deformations. Improved solutions can be obtained by adding a series expansion in terms of eigenwarpings to the uncorrected solution. The present theory is used to investigate the stresses and displacements of a cantilevered, rectangular box curved beam subjected to a uniformly distributed load. It is observed that the numerical results obtained agree well with the data from FEM.

Keywords: naturally curved and twisted beam, small initial curvature, small displacement theory, eigenwarping, generalized warping coordinate

0 INTRODUCTION

Static and dynamic analysis of naturally curved and twisted beams with closed thin-walled cross sections has many important applications in mechanical, civil and aeronautical engineering due to their outstanding engineering properties, such as streamlined modeling and favorable loaded characteristics. The structural behavior of the beams is no longer appropriately modeled with the classical beam theory ([1] to [3]), and a more advanced theory is much needed to overcome the demerits of the classical beam theory. Much research has been done in the theories for straight beams and curved beams ([4] to [14]), however, much less has been done for naturally curved and twisted beams. Bauchau and Bauchau et al. ([15] and [16]) provided a comprehensive treatment to the problem of warping using variational principles to model thin-walled straight beams made of anisotropic materials, however, their modes can not be used for naturally curved and twisted beams straightforwardly. Based on small displacement theory, the main contribution of the present work is to derive a set of orthonormal eigenwarpings and equivalent constitutive equations that can be used for the analysis of naturally curved and twisted beams. In addition, the correction to transverse shear deformations is also included in the present formulations.

1 GEOMETRY AND CONSTITUTIVE RELATIONS OF THE BEAM

Let the locus of the cross-sectional centroid of the beam be a continuum curve \( l \) in space, the tangential, normal and bi-normal unit vectors of the curve are \( t \), \( n \) and \( b \), respectively. The Frenet-Serret formula, for a smooth curve, is:

\[
\frac{d}{ds} t = k_1 n, \quad \frac{d}{ds} n = -k_1 t + k_2 b, \quad \frac{d}{ds} b = -k_2 n,
\]

where \( \cdot' \) means derivative with respect to \( s \), \( s \), \( k_1 \) and \( k_2 \) are arc coordinate, curvature and torsion of the curve, respectively.

We introduce \( \xi \) - and \( \eta \) - directions in coincidence with the principal axes through the centroid \( O_1 \), as shown in Fig. 1. The angle between the \( \xi \) - axis and normal \( n \) is represented by \( \theta \), which is generally a function of \( s \). If the unit vectors of \( O_1 \xi \) and \( O_1 \eta \) are represented by \( i_\xi \) and \( i_\eta \), then:

\[
i_\xi = n \cos \theta + b \sin \theta
\]

\[
i_\eta = -n \sin \theta + b \cos \theta.
\]

From Eq. (1) the following expressions are obtained:

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Yu, A. - Yang, R. - Hao, Y.

\[ t' = k_1 i_z - k_2 i_\eta \]
\[ i_\eta' = -k_1 t + k_1 i_\eta \]
\[ i_z' = k_2 t - k_1 i_z \]

which \( k_1 = k_1 \sin \theta \), \( k_\eta = k_1 \cos \theta \), \( k_z = k_2 + \theta' \). (3)

**2 INTERNAL FORCES, EQUILIBRIUM EQUATIONS AND KINEMATIC EQUATIONS**

Simplifying stress vectors to the centroid \( O_1 \) on the cross section \( A \), the principal vector \( Q \) and principal moment \( M \) can be obtained, of which components are respectively denoted by \( Q_1, Q_\eta, Q_z \) and \( M_s, M_\eta, M_z \) so:

\[ Q = Q_1 t + Q_\eta i_\eta + Q_z i_z, \]
\[ M = M_s t + M_\eta i_\eta + M_z i_z, \]

where \( Q_1 \) is axial force, \( Q_\eta \) and \( Q_z \) are shear forces, \( M_s \) is torque, \( M_\eta \) and \( M_z \) are bending moments, as shown in Fig. 3. The external forces and moments per unit length along the axis of the beam are indicated by \( p \) and \( m \) as:

\[ p = p_s t + p_\eta i_\eta + p_z i_z, \]
\[ m = m_s t + m_\eta i_\eta + m_z i_z. \]

The equilibrium equations are:

\[ \frac{d}{ds} \{Q\} - [K] \cdot \{Q\} + \{p\} = \{0\}, \]
\[ \frac{d}{ds} \{M\} - [K] \cdot \{M\} - [H] \cdot \{Q\} + \{m\} = \{0\}, \]

where

\[ \{Q\} = [Q_1, Q_\eta, Q_z]^T, \{M\} = [M_s, M_\eta, M_z]^T, \]
\[ \{p\} = [p_s, p_\eta, p_z]^T, \{m\} = [m_s, m_\eta, m_z]^T, \]
\[ [K] = \begin{bmatrix} 0 & k_\eta & -k_z \\ -k_\eta & 0 & k_s \\ -k_z & -k_s & 0 \end{bmatrix}, [H] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}. \]

The general solutions are [17]:

**Fig. 1. Geometry of the beam**

**Fig. 2. Closed cell thin-walled beam model**

**Fig. 3. Stress resultants developed on a typical beam element**
\[ Q = [A] \left( \{ Q_s \} + \int [A]^T \cdot \{ p \} ds \right), \]
\[ M = [A] \left( \{ M_s \} + \int [A]^T \cdot ([H] \cdot [A]) \right), \]
\[ \left( \{ Q_s \} + \{ Q^* \} \right) - \{ m \} \left( ds \right), \]
where \( \{ Q_s \} \) and \( \{ M_s \} \) are integration constants,
\[ \{ Q^* \} = - \int [A]^T \cdot \{ p \} ds. \]

If the base vectors of special fixed right-handed rectangular coordinate system are \( i_x, i_y, i_z \), then:
\[ [A] = \begin{bmatrix} t \cdot i_x & t \cdot i_y & t \cdot i_z \\ i_x \cdot i_x & i_y \cdot i_y & i_z \cdot i_z \\ i_y \cdot i_x & i_y \cdot i_y & i_y \cdot i_z \\ i_z \cdot i_x & i_z \cdot i_y & i_z \cdot i_z \end{bmatrix}. \]

The kinematic equations are:
\[ e_x = u_x' - k_1 u_x + k_2 u_y, \]
\[ e_y = u_y' + k_2 u_x - k_1 u_y - \phi_y, \]
\[ e_z = u_z' - k_1 u_z + k_2 u_y + \phi_y, \]
\[ \omega_x = \phi_x' - k_1 \phi_x + k_2 \phi_y, \]
\[ \omega_y = \phi_y' - k_1 \phi_y + k_2 \phi_y, \]
\[ \omega_z = \phi_z' - k_1 \phi_z + k_2 \phi_y, \]
(8)
where \( e_x, e_y, e_z, \omega_x, \omega_y, \omega_z \) are respectively generalized strains corresponding to the internal forces \( Q_x, Q_y, M_x, M_y, M_z \) and \( u_x, u_y, u_z, \phi_x, \phi_y, \phi_z \), \( \omega_x, \omega_y, \omega_z \) are the displacement components corresponding to the loads \( p_x, p_y, m_x, m_y, m_z \). The boundary conditions should be given by the following prescribed qualities:
\[ Q_x \text{ or } u_x, Q_y \text{ or } u_y, Q_z \text{ or } u_z, \]
\[ M_x \text{ or } \phi_x, M_y \text{ or } \phi_y, M_z \text{ or } \phi_z, \]
(9)

Eqs. (8) can be rewritten as:
\[ \frac{d}{ds} \{ \varphi \} - [K] \{ \varphi \} - \{ \omega \} = \{ 0 \}, \]
\[ \frac{d}{ds} \{ u \} - [K] \{ u \} - [H] \{ \varphi \} - \{ \varepsilon \} = \{ 0 \}, \]
(10)
where:
\[ \{ \varphi \} = \begin{bmatrix} \varphi_x & \varphi_y & \varphi_z \end{bmatrix}^T, \quad \{ u \} = \begin{bmatrix} u_x & u_y & u_z \end{bmatrix}^T, \]
\[ \{ \varphi \} = \begin{bmatrix} \varphi_x & \varphi_y & \varphi_z \end{bmatrix}^T, \quad \{ u \} = \begin{bmatrix} u_x & u_y & u_z \end{bmatrix}^T, \]
\[ \{ \omega \} = \begin{bmatrix} \omega_x & \omega_y & \omega_z \end{bmatrix}^T, \quad \{ \varepsilon \} = \begin{bmatrix} \varepsilon_x & \varepsilon_y & \varepsilon_z \end{bmatrix}^T, \]
so the general solutions to the kinematic equations are [17]:
\[ \{ \varphi \} = [A] \left( \{ \varphi_0 \} + \{ \varphi^* \} \right), \]
\[ \{ u \} = [A] \left( \{ U_0 \} + \int [A]^T \cdot \{ \varepsilon \} + [H] \cdot [A] \right), \]
\[ \left( \{ \varphi_0 \} + \{ \varphi^* \} \right) \left( ds \right), \]
in which \( \{ \varphi_0 \} \) and \( \{ U_0 \} \) are integration constants,
\[ \{ \varphi^* \} = \int [A]^T \cdot \{ \omega \} ds. \]

3 STRUCTURAL ANALYSIS BY THE EIGENWARPING APPROACH

Eigenwarping theory is in fact to transform the solution of out-of-plane torsional warping of the cross-section into a problem for finding the eigenvalues and eigenvectors. The eigenvalue problem can be solved using a finite element technique where the eigenwarping function is discretized over the section of the beams. The solution to the problem will be improved by adding a series expansion in terms of eigenwarpings to the uncorrected solution. In addition, the correction to transverse shear deformations is also included in the present formulations.

Assuming that the deformations of the beam consist of stretching, bending and torsion, then the displacement field can be written as follows:
\[ u = Wt + U_1 + VI_0, \]
in which:
\[ W = u_1(s) + \eta \varphi_x(s) - \xi \varphi_y(s), \]
\[ U = u_x(s) - \eta \varphi_x(s), \quad V = u_y(s) + \xi \varphi_y(s). \]
(12)

The strain-displacement relations are [1]:
\[ \sqrt{e_{11}} = e_x + \eta \omega_x - \xi \omega_y, \]
\[ 2\sqrt{e_{12}} = e_{xy} - \eta \omega_y, \]
\[ 2\sqrt{e_{13}} = e_y + \xi \omega_x, \]
(13)
where \( e_x, e_y, e_{xy}, \omega_x, \omega_y, \omega_{xy} \) are the same as Eq. (8). For simplicity, the initial curvature \( k_1 \) is assumed small to assure that:
\[ \sqrt{e} = 1. \]

The assumption is realistic for most practical applications, hence it does not seriously
restrict the applicability of this model. In this development a set of orthonormal eigenwarpings that can be used for naturally curved and twisted beams is derived. An unloaded beam is now considered (i.e. \( p = m = 0 \)) and a solution of the following form is assumed ([1] and [15]):

\[
W(\zeta, s) = \varphi(\zeta) \alpha(s),
\]

\[
\varepsilon_\xi(s) = \bar{U} \alpha(s),
\]

\[
\varepsilon_\eta(s) = \bar{V} \alpha(s),
\]

\[
\omega_\alpha(s) = \bar{Z} \alpha(s),
\]

where \( \varphi(\zeta) \) and \( \alpha(s) \) are the eigenwarping modes of the cross-section and the generalized warping coordinates, respectively, and \( \bar{U}, \bar{V} \) and \( \bar{Z} \) are three unknown parameters. Substituting Eqs. (14) into the strain-displacement relations in [1], we obtain:

\[
e_{11} = \varphi \alpha'(s) + k_1 \left[ \frac{\partial \varphi}{\partial \zeta} \eta - \left( \frac{\partial \varphi}{\partial \eta} \right) s \right] \alpha(s),
\]

\[
\gamma = 2e_{11} \frac{d\xi}{d\zeta} + 2e_{13} \frac{d\eta}{d\zeta} = \bar{U} \alpha(s) \frac{d\xi}{d\zeta} - \eta \bar{Z} \alpha(s) \frac{d\xi}{d\zeta} + \left[ \frac{\partial \varphi}{\partial \zeta} \right] k_1 \varphi \alpha(s) \frac{d\xi}{d\zeta} + \bar{V} \alpha(s) \frac{d\eta}{d\zeta} + \bar{Z} \alpha(s) \frac{d\eta}{d\zeta} + \left[ \frac{\partial \varphi}{\partial \eta} \right] - k_1 \varphi \alpha(s) \frac{d\eta}{d\zeta} = \left( \frac{d\varphi}{d\zeta} - k_1 \varphi \frac{d\eta}{d\zeta} + k_1 \varphi \frac{d\xi}{d\zeta} \right) \alpha(s),
\]

\[
\left( \frac{d\varphi}{d\zeta} - k_1 \varphi \frac{d\eta}{d\zeta} + k_1 \varphi \frac{d\xi}{d\zeta} \right) \alpha(s) = \left( \frac{d\varphi}{d\zeta} - k_1 \varphi \frac{d\eta}{d\zeta} + k_1 \varphi \frac{d\xi}{d\zeta} \right) \alpha(s) + \bar{U} \frac{d\xi}{d\zeta} + \bar{V} \frac{d\eta}{d\zeta} + r \bar{Z} \alpha(s),
\]

where the variables are separated, and \( r \) is the distance from the centroid \( O_1 \) to the tangent to the cross-sectional curve (see Fig. 2). The total strain energy in the beam is there:

\[
\Pi = \frac{1}{2} \int \left( E \varepsilon_{11}^2 + G \gamma^2 \right) d\zeta ds,
\]

where the product of the generalized warping coordinate and its derivative can be eliminated according to small displacement theory, and the differential equation defining generalized warping coordinates and associated eigenvalues can be derived by minimizing with respect to \( \varphi, \bar{U}, \bar{V} \) and \( \bar{Z} \). The derivation is similar to that in [15]. The associated eigenvalues \( \mu_1^2 \) can also be written in the form of a Rayleigh quotient:

\[
\mu_1^2 = \left( \frac{\int \left( E \varphi \alpha \frac{d\varphi}{d\zeta} \right) \eta - \frac{\partial \varphi}{\partial \eta} \xi \right)^2}{\int \varphi \alpha \frac{d\varphi}{d\zeta} \zeta^2}.
\]
where \( e_{11}, e_{120r}, e_{130r} \) coincides with Eq. (13). The strains corresponding to this displacement field can be used to evaluate the total potential energy \( \Pi \):

\[
\Pi = \frac{1}{2} \int (Ee^2 + Gt^2) d\zeta ds
- \int (p_x u_x + p_y u_y + m_\varphi) ds,
\]

and taking into account the orthonormality relationships (17) it reads:

\[
\Pi = \Pi_{or} + \sum_i \int \left[ \frac{1}{2} \left( \alpha_i^2 + \mu_i^2 \alpha_i^2 \right) - d_i \right] d\zeta ds,
\]

where \( \Pi_{or} \) is the total potential energy for the original problem, and:

\[
d_i = Q_i (\overline{U}_i + k_\varphi \varphi) + Q_i (\overline{V}_i - k_\varphi \varphi) + M_i \overline{\varphi}.
\]

Eq. (21) shows the solutions to be decoupled from the corrective terms \( \alpha_i \) and the different corrective terms decoupled from each other. Minimizing \( \Pi_{or} \) will render the equilibrium Eq. (5) described with the internal forces and minimizing the corrective terms with respect to \( \alpha_i \) gives:

\[
\alpha_i^* - \mu_i^* \alpha_i = -d_i.
\]

This differential equation is solved readily and yields the solution of the problem as a series expansion of eigenwarps:

\[
W = W_{or} + \sum_i \varphi_i \alpha_i, \quad \varepsilon_i = \varepsilon_{or} + \sum_i \overline{U}_i \alpha_i, \\
\varepsilon_\varphi = \varepsilon_{or} + \sum_i \overline{V}_i \alpha_i, \quad \omega_\varphi = \omega_{or} + \sum_i \overline{\varphi} \alpha_i, \\
n = n_{or} + \sum_i E_i \varphi_i \alpha_i', \\
q = q_{or} + \sum_i G_i \left( \frac{d\varphi_i}{d\zeta} - k_\varphi \varphi \right) \frac{d\eta_i}{d\zeta} + k_\varphi \varphi \frac{d\zeta}{d\zeta} + \overline{V}_i \frac{d\zeta}{d\zeta} + \overline{V}_i \frac{d\eta_i}{d\zeta} + \overline{\varphi} \alpha_i(s),
\]

(24)

where \( n_{or} \) and \( q_{or} \) are the uncoupled solutions. When infinite series are used, Eq. (24) give the theoretical solution of the problem under the assumption of infinite in-plane rigidity of the section.

Introducing the internal forces defined by:

\[
Q_i = \int n d\zeta, \quad M_i = \int q r d\zeta, \\
Q_\varphi = \int q \frac{d\varphi}{d\zeta} d\zeta, \quad M_\varphi = \int q \frac{d\eta}{d\zeta} d\zeta, \\
Q_{or} = \int q \frac{d\zeta}{d\zeta} d\zeta, \quad M_{or} = -\int n \zeta d\zeta.
\]

(25)

In view of Eqs. (4), (19) and (25), noting that:

\[
\int n d\zeta = \int q d\zeta = \int q \frac{d\zeta}{d\zeta} d\zeta = 0,
\]

we obtain the equivalent constitutive equations described with the generalized strains and generalized warping coordinates:

\[
Q_i = S\varepsilon_i + \sum_i \int E_i \varphi_i d\zeta \alpha_i', \\
Q_\varphi = \int G_i A_i \overline{U}_i + G_i A_i \overline{V}_i + \int G_i A_i \overline{\varphi} \alpha_i, \\
Q_{or} = \int G_i A_i \overline{U}_i + G_i A_i \overline{V}_i + \int G_i A_i \overline{\varphi} \alpha_i
\]

(26)
Fig. 4). In this case of a curved, thin-walled rectangular beam (see applied to compute the stresses and displacements in previous sections, which will be directly demonstrate the theoretical formulations derived

\[ M_z = I_z \omega_z + \int_c G r \frac{d \theta_z}{d \zeta} d \zeta \zeta_{\omega_z} \]

\[ M_z = I_z \omega_z + \sum_i \int_c E \eta_i \varphi_i d \zeta \zeta_{\omega_z} \]

\[ M_z = I_z \omega_z - \sum_i \int_c E \xi_i \varphi_i d \zeta \zeta_{\omega_z} \]

\[ \beta = \frac{s}{a}, \quad k_\eta = k_\xi = \frac{1}{a} \]

\[ x = a \sin \beta, \quad y = a(1 - \cos \beta). \]

Using Eqs. (6) and (11), we have:

\[ M_z = M_{u_s} \cos \beta + M_{d_s} \sin \beta \]

\[ + Q_{s_0} a(1 - \cos \beta) + p_s a^2 (\sin \beta - \beta), \]

\[ M_z = -M_{u_s} \sin \beta + M_{d_s} \cos \beta \]

\[ + Q_{s_0} a \sin \beta - p_s a^2 (1 - \cos \beta), \]

\[ Q_{s} = Q_{s_0} - p_s s, \]

\[ \varphi_s = \varphi_{s_0} \cos \beta + \varphi_{s_0} \sin \beta \]

\[ + \cos \beta \int_0^\beta (\omega_s \cos \beta - \omega_s \sin \beta) d \beta \]

\[ + \sin \beta \int_0^\beta (\omega_s \sin \beta + \omega_s \cos \beta) d \beta, \]

\[ \varphi_s = -\varphi_{s_0} \sin \beta + \varphi_{s_0} \cos \beta \]

\[ - \sin \beta \int_0^\beta (\omega_s \cos \beta - \omega_s \sin \beta) d \beta \]

\[ + \cos \beta \int_0^\beta (\omega_s \sin \beta + \omega_s \cos \beta) d \beta, \]

\[ u_\eta = U_{s_0} + \varphi_{s_0} y - \varphi_{s_0} x + \int_0^\beta \varphi_i d \beta \]

\[ + \int_0^\beta \sin \beta \int_0^\beta (\omega_s \cos \beta - \omega_s \sin \beta) d \beta \]

\[ - \alpha \cos \beta \int_0^\beta (\omega_s \sin \beta + \omega_s \cos \beta) d \beta d \beta, \]

where \( G_z \) and \( G_\eta \) are the shear coefficients in \( \zeta \) - and \( \eta \) - directions for thin-walled beams \([18], S = \int_c \eta_\zeta \zeta d \zeta \) is the axial stiffness,

\[ I_{zz} = \int_c \eta_\zeta \zeta d \zeta \]

is the bending stiffness (similar definition for \( I_{\eta\eta} \),

\[ A_{zz} = \int_c G \left( \frac{d \theta_z}{d \zeta} \right)^2 d \zeta \]

is the shear stiffness (similar definitions for \( A_{\eta\eta} \) and \( A_{zz} \)), and

\[ I_{\eta\eta} = \int_c \eta_\zeta \zeta d \zeta \]

is the torsional stiffness.

It is observed that the internal forces of the beam depend on not only the generalized strains but also the eigenwarping functions, generalized warping coordinates and their derivatives.

5 EXAMPLE - A CURVED BEAM UNDER A UNIFORMLY DISTRIBUTED LOAD

Some numerical results are given to demonstrate the theoretical formulations derived in previous sections, which will be directly applied to compute the stresses and displacements of a curved, thin-walled rectangular beam (see Fig. 4). In this case \( \theta, k_\zeta \) and \( k_\eta \) in Eq. (3) are zero and \( k_\zeta \) is \( 1/R \). Fix the origin of the rectangular coordinate system at the end of the beam \(( s = 0)\), the axis of the beam being on the plane \( Ox \). The load acting is

\[ \{ m \} = [0 \quad 0 \quad 0]^T, \quad \{ p \} = [0 \quad 0 \quad 0]^T. \]

If the axis of the beam is a circle with radius \( \alpha \), one has

\[ \beta = \frac{s}{a}, \quad k_\eta = k_\xi = \frac{1}{a}, \]

\[ x = a \sin \beta, \quad y = a(1 - \cos \beta). \]

Using Eqs. (6) and (11), we have:

\[ M_z = M_{u_s} \cos \beta + M_{d_s} \sin \beta \]

\[ + Q_{s_0} a(1 - \cos \beta) + p_s a^2 (\sin \beta - \beta), \]

\[ M_z = -M_{u_s} \sin \beta + M_{d_s} \cos \beta \]

\[ + Q_{s_0} a \sin \beta - p_s a^2 (1 - \cos \beta), \]

\[ Q_{s} = Q_{s_0} - p_s s, \]

\[ \varphi_s = \varphi_{s_0} \cos \beta + \varphi_{s_0} \sin \beta \]

\[ + \cos \beta \int_0^\beta (\omega_s \cos \beta - \omega_s \sin \beta) d \beta \]

\[ + \sin \beta \int_0^\beta (\omega_s \sin \beta + \omega_s \cos \beta) d \beta, \]

\[ \varphi_s = -\varphi_{s_0} \sin \beta + \varphi_{s_0} \cos \beta \]

\[ - \sin \beta \int_0^\beta (\omega_s \cos \beta - \omega_s \sin \beta) d \beta \]

\[ + \cos \beta \int_0^\beta (\omega_s \sin \beta + \omega_s \cos \beta) d \beta, \]

\[ u_\eta = U_{s_0} + \varphi_{s_0} y - \varphi_{s_0} x + \int_0^\beta \varphi_i d \beta \]

\[ + \int_0^\beta \sin \beta \int_0^\beta (\omega_s \cos \beta - \omega_s \sin \beta) d \beta \]

\[ - \alpha \cos \beta \int_0^\beta (\omega_s \sin \beta + \omega_s \cos \beta) d \beta d \beta, \]

where \( M_{u_s}, M_{d_s}, Q_{s_0} \) are the values of \( M_s, M_d, Q_s \), at the end \( s = 0 \), and \( \varphi_{s_0}, \varphi_{s_0}, U_{s_0} \) are the values of \( \varphi_s, \varphi_s, u_\eta \) at \( s = 0 \), \( \omega_s, \alpha_s \), and \( \alpha_s \) are described with \( M_s, M_d, Q_s, \alpha_s \), and \( \alpha_s' \) by Eqs. (26).

The curved beam with radius \( \alpha = 400 \) mm is assumed to be fixed at one end \(( s = 0)\) and free at the other \(( s = l)\). Fig. 4b illustrates the cross-section at the free end of the beam. The following properties of this material are:

\[ E = 2.106 \times 10^5 \text{ MPa}, \quad G = 81.6 \times 10^3 \text{ MPa}. \]

The first step is to calculate the eigenwarpings \( \varphi_i \) and the associated eigenvalues \( \mu_i \). The eigenvalue problem (16) can be solved using a finite element technique. For this example, 43 nodal points are used to model the section and 24 eigenwarpings are extracted.
Fig. 4. Axis and cross-section at the free end of a plane curved beam

The bending and twisting behavior of the beam subjected to a uniformly distributed load $p_\eta$ in the $\eta$ direction will be described. In this case the differential Eq. (23) to be solved for improved solution is:

$$2 \left( \frac{\cos \beta}{\alpha'_{2}} \right) = \frac{2 \pi}{\alpha'_{1}} \frac{\cos \beta}{\alpha'_{2}}$$

The solution to Eq. (28) must be:

$$\alpha'_{1} = C_{s} \mu s + C_{h} \mu a \alpha + \alpha'_{s},$$

in which $\alpha'_{s}$ is a particular solution to Eq. (28). The complete solution of Eq. (28) becomes:

$$\alpha'_{1} = \frac{1}{\mu_{s}} p_{\eta} a \bar{V} \left( \frac{\pi}{\alpha'_{2}} - \beta \right)$$

$$+ \frac{1}{\mu_{1}} p_{\eta} a \bar{\Xi} \left( \frac{\pi}{\alpha'_{2}} - \beta \right)$$

$$- \frac{1}{\left( \alpha'_{2} + \mu_{s} \right)} p_{\eta} a \bar{\Xi} \cos \beta .$$

The boundary conditions are:

$$s = 0 (\beta = 0), \quad U_{u_{x}} = U_{b_{y}} = U_{b_{\eta}} = 0,$$

$$\varphi_{u_{x}} = \varphi_{b_{y}} = \varphi_{b_{\eta}} = 0, \quad \alpha_{i} = 0,$$

$$s = l (\beta = \beta_{i}), \quad M_{x} = M_{y} = Q_{\eta} = 0,$$

$$\alpha'_{i} = 0,$$

where $l = \pi \alpha'_{2}$, the integration constants determined by the aforementioned conditions are:

$$M_{u_{x}} = p_{\eta} a^{2} \left( \frac{\pi}{\alpha'_{2}} - 1 \right),$$

$$M_{b_{y}} = -p_{\eta} a^{2},$$

$$Q_{b_{\eta}} = \frac{\pi}{2} p_{\eta} a,$$

$$C_{1} = \left[ -\frac{1}{2} \left( 1 + \frac{\alpha'_{1}}{\mu_{s}} \right) \mu_{s} - 2 \frac{1}{\alpha'_{2}} \frac{\alpha'_{1}}{\mu_{s}^{2}} \cos \beta \right] p_{\eta},$$

$$C_{2} = \left[ \frac{1}{2} e^{\frac{\alpha'_{1}}{\mu_{s}}} \times \right.$$

$$\left. \mu_{s} \left( 1 + \frac{\alpha'_{1}}{\mu_{s}} \right) \right]$$

$$\left. - \frac{1}{2} e^{\frac{\alpha'_{1}}{\mu_{s}}} \right] p_{\eta} .$$

So far, the solutions to this problem have been obtained.

The determination of stresses and displacements is a significant problem in the static analysis of thin-walled beams. Let $V_{\lambda}$ present the displacement in the $\eta$ - direction of point $A$ on the cross section ($\beta = \pi/2$) of the beam due to the load $p_{\eta}$ shown in Fig. 4b, and $\varphi_{b}$ presents the tip twist angle of the beam. Theoretical results for $V_{\lambda}$ and $\varphi_{b}$ are obtained and compared with a 2-D finite element analysis (referred as the FEM results), according to the ANSYS program. To analyze the beam shown in Fig. 4 by the finite element method, we partition it into 3600 shell elements (SHELL 92), and the total number of nodal points is 10880. These cases for $p_{\eta}$ (0 $\rightarrow$ 1000 N/m) are shown in Fig. 5a and b. It is evident that the theoretical results are close to the data from FEM.
It is also interesting to compute the stress distributions of the beam. Fig. 6 shows the distributions of the axial stress flow $n$ in the upper face at the root ($\beta = 0$) and the shear stress flow $q$ in the upper face at cross section ($\beta = 45^\circ$) of the beam under a uniformly distributed load $p_n = 1000\, \text{N/m}$.

6 CONCLUSIONS

An analysis method for determining the stresses and displacements of naturally curved and twisted beams with closed thin-walled cross sections is developed based on small displacement theory. The effects of torsion-related warping, transverse shear deformations and extension-shearing coupling are included in the proposed model. The above results clearly indicate that the key factor to improving the stress and displacement predictive capability of a theory is to account for these non-classical effects correctly and find the eigenwarping modes of the cross-sections. The eigenwarpings depend on not only the material properties and geometry of the sections, but also the initial curvature and torsion of the beams. The theory suggested in this paper is not limited to thin-walled box beams. In the case of solid cross sections, the concept of eigenwarpings can be extended as long as the basic assumption remains valid, i.e., as long as the section can be assumed infinitely rigid in its own plane.

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7 REFERENCES


