Strojniški vestnik - Journal of Mechanical Engineering 55(2009)12, 749-754
UDC 532.5

Paper received: 26.04.2007
Paper accepted: 25.09.2009

An Analytical Solution of the Navier-Stokes Equation for Flow over a Moving Plate Bounded by Two Side Walls

M. Emin Erdoğan - C. Erdem İmrak*
Istanbul Technical University, Faculty of Mechanical Engineering, Turkey

An exact solution of the Naiver-Stokes equation for unsteady flow over a moving plate between two side walls is given. This solution solves the problem that arises calculating shear stress at the bottom wall when the expression of velocity presented in literature is used. The variation of the shear stress at the bottom wall with the distance between two side walls for various values of the non-dimensional time is illustrated and it is shown that when the value of non-dimensional time is equal to unity, the shear stress approaches the asymptotic value. Furthermore, the volume flux across a plane normal to the flow is calculated and it is found that when the value of the non-dimensional time is equal to unity, the volume flux approaches the asymptotic value.

Keywords: Naiver-Stokes equation, analytical solution, sine transform, steady flow, unsteady flow

0 INTRODUCTION

An exact solution of the Navier-Stokes equation for unsteady flow over a plane wall and between two side walls perpendicular to the plane is given. Obtaining the exact solutions of the Navier-Stokes equation is very important because they provide a standard for checking the accuracies of many approximate numerical or empirical methods. Furthermore, the accuracies of the boundary layer approximation and the slow flow approximations can be tested by using the exact solutions of the Navier-Stokes equation. Although computer techniques make the complete integration of the Navier-Stokes equation feasible, the accuracy of the results can be established by a comparison to an exact solution. The exact solution given in this paper is for an unsteady flow in a viscous fluid generated by a plane wall moved suddenly and by two side walls held stationary. In the absence of the side walls, the flow reduces to the flow caused by a plane wall moved suddenly, which is termed Stokes problem or Rayleigh-type flow.

The steady flow over a plane wall moving at a constant speed between two side walls was considered in [1] to present a simple model of a paint-brush due to Taylor. The mathematical problem is of the familiar boundary value form. The solution which satisfies the boundary conditions at the bottom wall, at the side walls and at infinity is given in the form of a series [1]. This series which gives the velocity of the steady state flow is a convergent series, but term-by-term differentiation of this series, which is used to calculate the shear stress at wall, does not give a convergent series. This situation arises not only due to the flow extending to infinity but also due to the flow bounded either by a horizontal plane at the top or by a horizontal free surface at the top [2]. A similar problem occurs in the conduction of heat [3], since the governing equation is the same for both problems.

Unsteady flows of both Newtonian and non-Newtonian fluids over a plane wall in the absence of the side walls have been investigated by many authors [4] to [11]. The aim of this paper is to investigate the unsteady flow of a viscous fluid generated by a plane wall between two side walls. By using the Laplace transform method, the velocity is given in a series form.

When the distance between two side walls increases, the expression of the velocity reduces to that of the velocity for an unsteady flow of a fluid generated by a plane wall moved suddenly. When non-dimensional time moves towards infinity, the expression of velocity approaches to that of the velocity for the steady flow.

The shear stress at the bottom wall cannot be calculated by the expression of the velocity obtained by using the Laplace transform method. For this reason, by using the sine transformation, another expression for velocity is obtained. At any time, shear stress at the bottom wall is at its minimum at the middle of the plane and increases towards the side walls. The volume flux across the plane normal to the flow is given in terms of a

*Corr. Author's Address: ITU Makina Fakultesi, Inonu Caddesi no. 67, Gumussuyu 34437, Istanbul, Turkey, imrak@itu.edu.tr

definite integral. This integral provides a new function which is defined by the integral [12].

1 UNSTEADY FLOW

The fluid is over a plane wall and between two side walls perpendicular to the plane. The side walls extend to infinity in the *x*- and *z*-direction as shown in Figure 1. The fluid is suddenly set in motion by moving the bottom wall at constant speed in the absence of an imposed pressure gradient. The governing equation is

$$\frac{\partial u}{\partial t} = \nu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right). \tag{1}$$

The boundary and the initial conditions are

$u(\pm b, z, t) = 0$, for all t,
$u(y, 0, t) = U$ for all $t > 0$ and $-b < y < b$,
$u(y, \infty, t) = 0$ for all t and $-b < y < b$,
$u(y, z, 0) = 0$ for $-b < y < b$ and $z > 0$. (2)

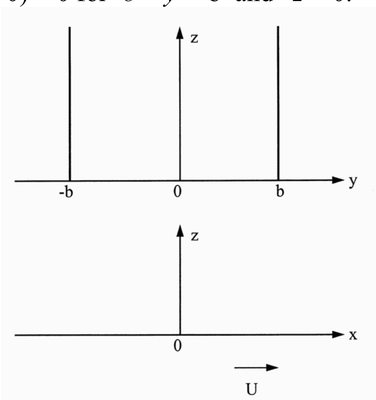

Fig. 1. *Flow geometry and coordinate system*

By using the Laplace transform method one finds [3]

$$\frac{u}{U} = \frac{2}{\pi}\sum_0 \frac{(-1)^n}{2n+1}\left[e^{-k_n z} erfc\left(\frac{z}{2\sqrt{\nu t}} - k_n\sqrt{\nu t}\right) + e^{k_n z} erfc\left(\frac{z}{2\sqrt{\nu t}} + k_n\sqrt{\nu t}\right)\right]\cos k_n y, \tag{3}$$

where $k_n = (2n + 1)\pi / 2b$ and

$$erfc\ x = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-\xi^2} d\xi$$

is the complementary error function [13]. When t goes to infinity the expression of the velocity given by Eq. (3) approaches the steady state which is given in the following form

$$\frac{u}{U} = \frac{4}{\pi}\sum_{n=0} \frac{(-1)^n}{2n+1} e^{-k_n z}\cos k_n y\,.$$

When z/b goes to infinity u/U becomes zero. In the absence of the side walls, namely, when b goes to infinity Eq. (3) takes the following form

$$\lim_{b\to\infty}\frac{u}{U} = erfc\frac{z}{2\sqrt{\nu t}}\,.$$

The variation of u/U at $y = 0$ with z/b for various values of $\nu t/b^2$ is illustrated in Fig. 2. For $\nu t/b^2 = 0.01$, z/b, is about 0.5 and for $\nu t/b^2 = 1$ the velocity approaches the asymptotic value.

The volume flux across the plane normal to the flow can be found in the following form

$$\frac{Q}{Ub^2} = \frac{32}{\pi^3}\sum_{n=0}(2n+1)^{-3} erf\left[\frac{\pi(2n+1)}{2}\sqrt{\frac{\nu t}{b^2}}\right]. \tag{4}$$

When t goes to infinity Eq. (4) reduces to

$$\frac{Q}{Ub^2} = \frac{32}{\pi^3}\sum_0 \frac{1}{(2n+1)^3}\,.$$

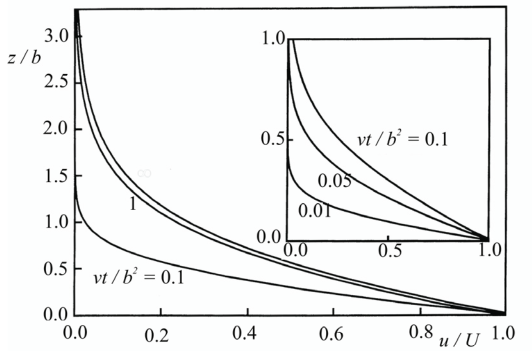

Fig. 2. *The variation of u/U at y* = 0 *with z/b for various values of* $\nu t/b^2$

The numerical value of the right hand side is about 1.0855. The flux corresponds to a mean thickness of the order 0.5427*b*. For $\nu t/b^2 = 1$, the volume flux approaches the asymptotic value.

The variation of the volume flux with $\nu t/b^2$ is illustrated in Fig. 3.

The shear stress at the bottom wall cannot be calculated by the expression of the velocity given by Eq. (3). For this reason, another expression for velocity by using the sine transformation is obtained. The solution of Eq. (1) can be written as

$$u = u_0(y, z) + u_1(y, z, t),$$

where u_0 and u_1 satisfy the following differential equations

$$\frac{\partial^2 u_0}{\partial y^2}+\frac{\partial^2 u_0}{\partial z^2}=0 \quad \text{and} \quad \frac{\partial u_1}{\partial t}=v\left(\frac{\partial^2 u_1}{\partial y^2}+\frac{\partial^2 u_1}{\partial z^2}\right),$$

u_0 shows the steady-state part and u_1 corresponds to the transient part. The boundary and initial conditions become

$u_0(\pm b, z) = 0$,
$u_0(y, 0) = U$,
$u_0(y, \infty) = 0$,
$u_1(\pm b, z, t) = 0$,
$u_1(y, 0, t) = 0$,
$u_1(y, \infty, t) = 0$,
$u_0(y, z) + u1(y, z, t) = 0$.

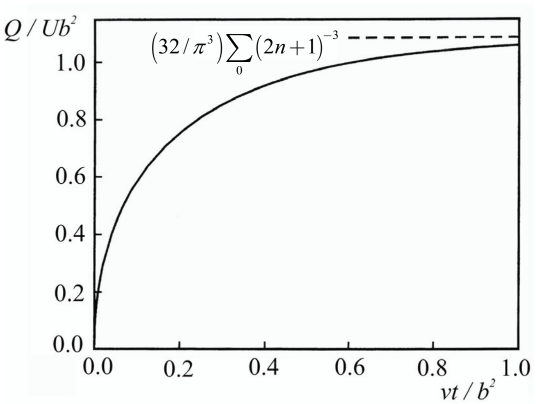

Fig. 3. *The variation of* Q/Ub^2 *with* vt/b^2

The sine transform of u_0 is [14]

$$\bar{u}_0=\int_0^\infty u_0 \sin\lambda z\, dz$$

and the boundary condition becomes

$$\bar{u}_0(\pm b)=0.$$

The problem is reduced to the solution of the following ordinary differential equation

$$\frac{d^2\bar{u}_0}{dy^2}-\lambda^2\bar{u}_0=-\lambda U.$$

The solution is

$$\bar{u}_0=\frac{U}{\lambda}\left(1-\frac{\cosh\lambda y}{\cosh\lambda b}\right).$$

The inverse of u_0 is given by the relation [14]

$$u_0=\frac{2}{\pi}\int_0^\infty \bar{u}_0 \sin\lambda z\, d\lambda,$$

and since

$$\int_0^\infty \frac{\sin\lambda z}{\lambda} d\lambda=\frac{\pi}{2},$$

u_0 can be written as

$$\frac{u_0}{U}=1-\frac{2}{\pi}\int_0^\infty \frac{\cosh\lambda y}{\lambda\cosh\lambda b}\sin\lambda z\, d\lambda \cdot$$

The sine transform of u_1 is

$$\bar{u}_1=\int_0^\infty u_1 \sin\lambda z\, dz,$$

and the conditions are

$$\bar{u}_1(\pm b,t)=0,$$
$$\bar{u}_0(y)+\bar{u}_1(y,0)=0.$$

The problem is reduced to the solution of the following partial differential equation

$$\frac{\partial\bar{u}_1}{\partial t}=v\frac{\partial^2\bar{u}_1}{\partial y^2}-v\lambda^2\bar{u}_1.$$

The boundary condition $\bar{u}_1(\pm b,t)=0$ suggests a solution in the following form

$$\bar{u}_1=\sum_{n=0} A_n e^{-v\left(k_n^2+\lambda^2\right)t}\cos k_n y,$$

where $k_n = (2n + 1)\pi/2b$ and A_n can be obtained by the equation

$$\frac{U}{\lambda}\left(1-\frac{\cosh\lambda y}{\cosh\lambda b}\right)+\sum_{n=0} A_n \cos k_n y=0.$$

This equation gives

$$A_n=-\frac{2(-1)^2 U}{bk_n}\frac{\lambda}{k_n^2+\lambda^2}.$$

The inverse of $\bar{u}_1$ is

$$\frac{u_1}{U}=\frac{-4}{\pi}\sum_{n=0}\frac{(-1)^n e^{-vk_n^2 t}\cos k_n y}{bk_n}\int_0^\infty \frac{\lambda e^{-v\lambda^2 t}\sin\lambda z}{k_n^2+\lambda^2}d\lambda.$$

Therefore, the application of the sine transform method to Eq. (1), gives

$$\frac{u}{U}=1-\frac{2}{\pi}\int_0^\infty \frac{\cosh\lambda y}{\lambda\cosh\lambda b}\sin\lambda z\, d\lambda$$
$$-\frac{4}{\pi}\sum_{n=0}\frac{(-1)^n e^{-k_n^2 vt}\cos k_n y}{bk_n}\int_0^\infty \frac{\lambda e^{-\lambda^2 vt}\sin\lambda z}{k_n^2+\lambda^2}d\lambda. \quad (5)$$

It can be shown that Eq. (5) satisfies Eq. (1). When t goes to infinity Eq. (5) reduces to the expression of the velocity in the case of the steady state in the following form

$$\frac{u}{U}=1-\frac{2}{\pi}\int_0^\infty \frac{\cosh\lambda y}{\lambda\cosh\lambda b}\sin\lambda z\, d\lambda,$$

or by using the integral

$$\int_0^\infty \frac{\cosh \lambda y}{\lambda \cosh \lambda b} \sin \lambda z \, d\lambda = \frac{\pi}{2} - \tan^{-1} \frac{\cos(\pi y/2b)}{\sinh(\pi z/2b)}$$

one finds

$$\frac{u}{U} = \frac{2}{\pi} \tan^{-1} \frac{\cos(\pi y/2b)}{\sinh(\pi z/2b)} .$$

When $t = 0$ Eq.(5) gives

$$1 - \frac{2}{\pi} \int_0^\infty \frac{\cosh \lambda y}{\lambda \cosh \lambda b} \sin \lambda z \, d\lambda =$$

$$\frac{4}{\pi} \sum_{n=0} \frac{(-1)^n}{(2n+1)} e^{-k_n z} \cos k_n y .$$

The expression of the velocity given by Eq. (5) satisfies the following conditions: when *z/b* goes to infinity *u/U* vanishes and in the absence of the side walls *u/U* reduces to the expression that is given by the Stokes problem. Eq. (5) can be written in the following form

$$\frac{u}{U} = \frac{2}{\pi} \tan^{-1} \frac{\cos(\pi y/2b)}{\sinh(\pi z/2b)}$$

$$- \frac{2}{\pi} \sum_{n=0} \frac{(-1)^n \cos k_n y}{2n+1} \left\{ 2 e^{-k_n z} \right.$$

$$- \left[e^{-k_n z} erfc\left(\frac{z}{2\sqrt{\nu t}} - k_n \sqrt{\nu t} \right) \right.$$

$$\left. \left. + e^{k_n z} erfc\left(\frac{z}{2\sqrt{\nu t}} + k_n \sqrt{\nu t} \right) \right] \right\} . \tag{6}$$

When *t* goes to zero, Eq. (6) gives

$$\tan^{-1} \frac{\cos(\pi y/2b)}{\sinh(\pi z/2b)} = 2 \sum_{n=0} \frac{(-1)^n e^{-k_n z}}{2n+1} \cos k_n y .$$

The volume flux across a plane normal to the flow is

$$Q = \frac{4Ub}{\pi} \int_0^\infty \left(1 - \frac{\tanh \lambda b}{\lambda b} \right) \frac{d\lambda}{b^2}$$

$$- \frac{8U}{\pi b} \sum_{n=0} \frac{1}{k_n^2} \int_0^\infty \frac{e^{-(\lambda^2 + k_n^2)\nu t}}{\lambda^2 + k_n^2} d\lambda ,$$

or by using the integral

$$\int_0^\infty \frac{e^{-(\lambda^2 + k_n^2)\nu t}}{\lambda^2 + k_n^2} d\lambda = \frac{\pi}{2k_n} erfc\left(k_n \sqrt{\nu t} \right) ,$$

one finds

$$\frac{Q}{Ub^2} = \frac{4}{\pi} F(\infty) - 4 \sum_{n=0} \frac{1}{b^3 k_n^3} erfc\left(k_n \sqrt{\nu t} \right) , \tag{7}$$

where $F(\infty)$ is given by an integral

$$F(\infty) = \int_0^\infty \frac{\xi - \tanh \xi}{\xi^3} d\xi$$

and the numerical value of e $F(\infty)$ is 1.0855090289 correct to ten decimal places [12]. If one uses the equality

$$F(\infty) = \frac{8}{\pi^3} \sum_{n=0} (2n+1)^{-3} ,$$

then, this expression of the volume flux reduces to Eq. (4).

The shear stress at the bottom wall can be calculated by Eq. (5). Taking the derivative of Eq. (5) with respect to *z* and then putting *z* = 0 one finds

$$(\sigma_{xz})_{z=0} = \mu \left(\frac{\partial u}{\partial z} \right)_{z=0} = - \frac{2\mu U}{\pi} \int_0^\infty \frac{\cosh \lambda y}{\cosh \lambda b} d\lambda$$

$$- \frac{4\mu U}{\pi} \sum_{n=0} \frac{(-1)^n \cos k_n y}{b k_n} \int_0^\infty \frac{\lambda^2 e^{-(\lambda^2 + k_n^2)t}}{\lambda^2 + k_n^2} d\lambda ,$$

and by using the integral

$$\int_0^\infty \frac{\lambda^2 e^{-(\lambda^2 + k_n^2)t}}{\lambda^2 + k_n^2} d\lambda = \frac{\pi}{2} \frac{1}{\sqrt{\nu t}} ierfc\left(k_n \sqrt{\nu t} \right) ,$$

where

$$i^n erfc\, x = \int_x^\infty i^{n-1} erfc \xi \, d\xi ,$$

$$i^0 erfc\ x = erfc\, x ,$$

are the integrals of the complementary error functions [13], one obtains

$$\frac{(\sigma_{xz})_{z=0}}{\mu U/b} = - \left[\frac{1}{\cos(\pi y/2b)} \right.$$

$$\left. + 2 \sum_{n=0} \frac{(-1)^n \cos k_n y}{k_n \sqrt{\nu t}} ierfc\, k_n \sqrt{\nu t} \right] . \tag{8}$$

When *t* goes to infinity, since *ierfc*(∞) = 0, Eq. (8) reduces to the expression for the steady state

$$\frac{(\sigma_{xz})_{z=0}}{\mu U/b} = - \frac{1}{\cos(\pi y/2b)} .$$

The variation of $(\sigma_{xz})_{z=0} / (\mu U/b)$ with *y/b* for various values of $\nu t/b^2$ is illustrated in Figure 4. When $\nu t/b^2$ = 1, the shear stress approaches the asymptotic value.

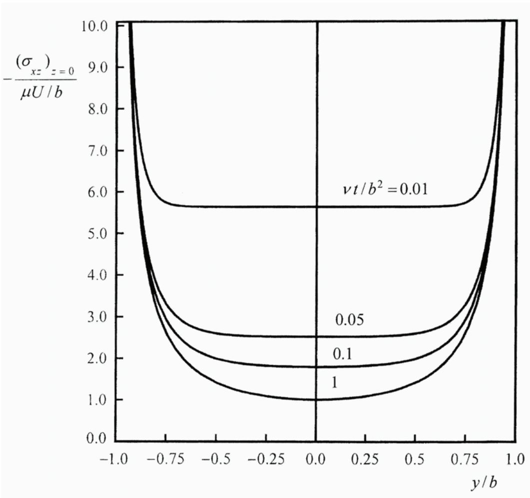

Fig. 4. *The variation of* $(\sigma_{xz})_{z=0}/(\mu U/b)$ *with y/b for various values of* $\nu t/b^2$

From Fig. 4 it can be seen that the shear stress at the bottom wall is at its minimum at the middle of the plate and increases towards the side walls and at the side walls, namely, $y=b$, since $\cos k_n b = 0$ and since [15]

$$\int_0^\infty \cos \lambda z d\lambda = \pi\delta(z) ,$$

where $\delta(z)$ is the Dirac delta function, becomes

$$\frac{(\sigma_{xz})_{y=b}}{\mu U/b} = -\pi\delta(z) .$$

It is can be clearly seen that this result is independent of time and it shows the corner singularity of the solution. This situtation is remarked in the second of the boundary conditions given by Eq. (2).

2 CONCLUSIONS

An analytical solution for an unsteady flow generated by a plane wall moved suddenly and by two side walls held stationary is given. In the absence of the side walls, the flow reduces to the flow caused by a flat plate moved suddenly, which is termed Stokes problem or Rayleigh-type flow. When non-dimensional time moves towards infinity the solution tends to that of the steady flow in the same geometry. The steady and the unsteady flows in the geometry considered are investigated to give a correct result for shear stress at the bottom wall. The shear stress at the bottom wall can be calculated from the velocity obtained by using the sine transformation. It is shown that the shear stress at the bottom wall is at its minimum at the middle of the plate and it increases towards the side walls. Furthermore, the volume flux across a plane normal to the flow is calculated and it is shown that when value of non-dimensional time is equal to unity, the volume flux approaches the asymptotic value.

4 REFERENCES

[1] Batchelor, G.K. (1967) An introduction to fluid dynamics, *Cambridge University Press*, Cambridge, ISBN 0521 66396 2.
[2] Erdoğan, M.E. (1997) Effects of the side walls on the flow over a moving plane wall, *Applied Mechanics and Engineering,* vol. 2, no. 3, p. 415-432.
[3] Carslaw, H.S., Jaeger, J.C. (1973) Conduction of heat in solid, 2nd Ed. *Oxford University Press*, Oxford.
[4] Erdoğan, M.E. (1997) Unsteady flow of a viscous fluid over a plane wall, *Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM)***77,** p. 733-740.
[5] Tokuda, N. (1968) On the impulsive motion of a flat plate in a viscous fluid, *Journal of Fluid Mechanics* 33**,** p. 657-672.
[6] Zeng, Y., Weinbaum, S. (1995) Stokes problem for moving half-plane, *Journal of Fluid Mechanics* 287**,** p. 59-74.
[7] Puri, P., Kulshrestha, P.K. (1976) Unsteady hydromagnetic boundary layer in a rotating medium, *Journal of Applied Mechanics, Transactions ASME* 43E**,** p. 205-208.
[8] Erdoğan, M.E. (2000) A note on an unsteady flow of a viscous fluid due to an oscillating plane wall, *International Journal of Non-Linear Mechanics* vol. 35, no. 2, p.1-6.
[9] Erdoğan, M.E. (2003) On unsteady motion of a second-order fluid over a plane wall, *International Journal of Non-Linear Mechanics* vol. 38, no. 7, p. 1045-1051.
[10] Puri, P. (1984) Impulsive motion of a flat plate in a Rivlin-Ericksen fluid, *Rheologica Acta* 23, p. 451-453.
[11] Bandelli, R., Rajagopal, K.R., Galdi, C.P. (1995) On some unsteady motions of fluids of second grade, *Archives of Mechanics* 47, p. 661-676.
[12] Erdoğan, M.E., Imrak, C.E. (2006) On the properties of a special function defined by an

integral, *Mathematical & Computational Applications* vol. 11, no. 1, p. 11-18.

[13] Abromowitz, M., Stegun, I.A. (Eds.) (1965) Handbook of mathematical functions, *Dover Publications*, New York, ISBN 486-61272-4.

[14] Tranter, C.J. (1951) Integral transforms in mathematical physics, *Methuen*, London.

[15] Robinson, R.M. (1970) The effects of a corner on a propagating internal gravity wave, *Journal of Fluid Mechanics* 42, p. 257-267.