STROJNIŠKI VESTNIK
MECHANICAL JOURNAL

VOL. 16 LJUBLJANA, JANUARY 1970 No. 1

ENGLISH TRANSLATION OF THE LEADING ARTICLE

UDC 624.072.2:681.32

Calculation of several times supported continuous beams by reduction method on digital computers*

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1. INTRODUCTION

The post-war period brought broad possibilities of contemporary handling of present problems and that particularly in the domain of engineering sciences. In this respect we are also conscious that we must introduce contemporary designing manners with today's calculation methods adapted to the application of digital computers.

Till now, it is known that in the domain of structural mechanics we have used very successfully various classical methods for handling statically indeterminate structures. If there are not available electronic computers, some of these methods are still very favourable e. g. the method of forces, the deformation method, Cross's, Kani's and Rosman's procedures. Nevertheless, the general progress of engineering already requires more rational handling of theoretical and applicative problems. Thus, we must prepare more or less known calculating methods for the use with computers. Moreover we must be acquainted with sufficient knowledge of the *matrix calculus* which facilitates representation of complicated mathematical operations by a group of algebraic or numerical quantities the simple form of which satisfies the designation of the whole group. Thereby we should not forget that the choice of the *numerical algorithm* for the solution of a given problem is very important.

Today, engineering-designing problems should be the main domain for the use of digital computers and this either with routine designing work or with scientific research work. It is known that the application of computers in science has given incredible results for the exploration of the universe. Anyhow, with our modest possibilities, at least in the initial development phase it would be very advantageous still to remain at terrestrial mechanics and to take in hand as successful as possible treatments of various problems of structural mechanics. Recently also in our country the number of digital computers has grown rapidly. Simultaneously with the development and installation of computers it is necessary to improve also calculating methods and their programming.

The entire procedure of treating scientific-technical problems comprises their formulations and all phases up to the interpretation of results which makes up an enough dynamic system. Thereby the fundamental work and the mutual interconnection of the calculating activity is put into the hands of *man* — the *searcher* and *programmer*, while the calculating and memory process is taken over by the computer itself. The division of duties between man and computer rests principally on the criterion that both participants be best engaged and also best utilized. For clearness of the entire calculating procedure, we prepare a *flow diagram* which shows the particular calculation phases and their sequence. Programming is the procedure of translating the mathematical problem, formulated in *mathematical language* into the *language of the digital computer*.

With the development of computers, there appear various programming languages which in general are very similar. Probably ALGOL 60 is most important for engineering and it is officially described by two *"Reports on the Algorithmic Language ALGOL 60"* by Backus (1960) and by Naur (1963). The next, most widespread in America and recently also in Europe is the FORTRAN (FOR-mula TRAN-slator) language, mostly used by the IBM. For special problems of structural mechanics, the Massachusetts Institute of Technology has elaborated a problem-orientation language STRESS (STR-uctural E-ngineering S-ystems S-olver).

These languages mentioned render possible the description of engineering problems by principles of numerical mathematics while the translation into a programme understandable for the machine is made by the compiler of the computer. The user or client for the solution of a given problem have usually scarcely an idea what it was necessary to do in order to make possible a successful use of the digital computer. From further deductions we shall ascertain that the automatic calculation of

* This article is a shortened wording of a treatise to be published in a special issue of "Zbornik Fakultete za strojništvo", (Yearbook of Faculty of Mechanical Engineering), Ljubljana, Yugoslavia.

digital computers warrants besides a great speed also the accuracy of mathematical operations from the preparation up to the final results. Simultaneously, static calculations made till now prove that the use of computers results in great time and cost savings with regard to classical methods.

2. GENERAL CALCULATING METHODS

From the definition of a statically indeterminate structure it results that for the treatment of indeterminate systems we must use besides equilibrium conditions still additionnal conditions resulting from elasticity of structures. Thereby it is our task to obtain the results required by the shortest possible way. Then we determine adequate dimensions of structures which satisfy their carrying capacity. Static problems of indeterminate structures can be generally treated with digital computers by the following calculation methods:

— reduction method,
— combined deformation and reduction method,
— method of stiffness matrices,
— method of forces,
— iteration methods.

For space limitations, this article will deal only with the reduction method which is treated in theoretical form together with an elaborated programme of calculation of continuous beams (arbitrarily supported, stiffly or elastically). A further article will explain the elaboration of programmes for the calculation of framed orthogonal plane frames and plane grids (e. g. an automobile chassis). Moreover some calculated characteristical examples are annexed having the aim to draw attention upon the possibility of use of prepared programmes for calculation on the digital computer Z-23 or on other computers in the frame of the computer centre at the University, Ljubljana.

3. REDUCTION METHOD

The essence of the reduction method consists in that the static state of the supporting element at the *beginning* of the field $(k + 1)$ is expressed by static quantities at the *end* of the field (k). The transfer or reduction of static quantities is realized only when the static state of one end (i. e. e. g. at the left side of the supporting structure) is known.

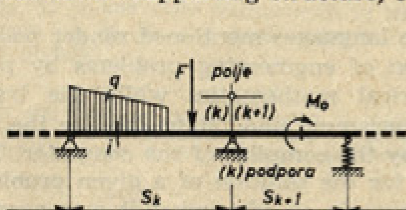


Fig. 1
polje = field, podpora = support

Their quantities are calculated from *boundary conditions* at the other end (i. e. e. g. on the right-hand side) and from *intermediate conditions* (i. e. somewhere between the left and right sides) of the given structure (Fig. 1).

In the reduction method we have departed from theoretical legalities and from matrix records of differential equations in a general form which facilitate the solution of any plane continuous beams, frame structures and plane grids.

3.1. Vector of state of static quantities

In a point i of the beam (Fig. 1), the state of static quantities is known when all six quantities are known:

u, w — displacement in direction of x or y axes,
φ — torsion ob beam axis with behalf to x axis,
M — bending moment,
Q — transverse force,
N — axial force.

These quantities of state are expressed by the vector of state η_i in form of a matrix with inscription (1).

$$\eta_i = \begin{bmatrix} u_i \\ w_i \\ \varphi_i \\ M_i \\ Q_i \\ N_i \\ 1 \end{bmatrix} \tag{1}$$

The seventh element of the matrix (1) which always equals *one*, takes account *of loadings* of the beam and of rules of multiplication of matrices. The given choice of directions of static quantities (Fig. 2) is advantageous for computer calculation. For drawing diagrams of axial forces N and bending moments M it is necessary to consider the classically introduced signs.

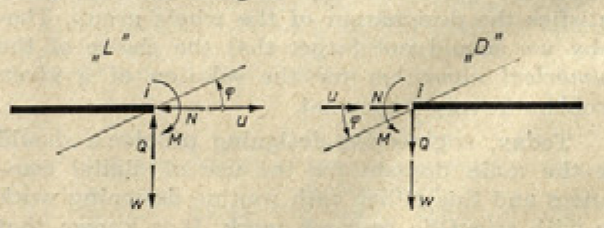


Fig. 2

3.2. Starting equations of reduction method

Quantities of state are calculated by differential equations

$$\frac{d^2M}{dx^2} = \pm\, q\,(x) \tag{2}$$

$$\frac{dN}{dx} = -\, p\,(x) \tag{3}$$

Considering the elasticity range in behaviour of the material we can write known expressions for the moment M and the axial force N

$$M = E\,J\,\frac{d^2 w}{dx^2} \tag{4}$$

$$N = \sigma\,.\,A = E\,\varepsilon\,A = E\,A\,\frac{du}{dx} \tag{5}$$

Moreover we shall treat problems of supporting structures having unvariable cross-sections along the axes of single beams. The sizes and shapes of cross-section can in a system of beams differ between them. From the quoted it follows that the starting equations (2) and (3) can take the following form:

$$\frac{d^2}{dx^2}\left(E\,J\,\frac{d^2 w}{dx^2}\right) = \pm\, q\,(x) \tag{6}$$

$$\frac{d}{dx}\left(E\,A\,\frac{du}{dx}\right) = -\,p\,(x) \tag{7}$$

where: $q\,(x)$ — vertical loading of beam in kp/m
$p\,(x)$ — horizontal loading of beam in kp/m
J — moment of inertia of cross-section in cm^4
A — cross-section of beam in cm^2.

3.2.1. *Integration of equations (6) and (7)*

a) For the integration of equation (6) the loading $q\,(x)$ must always be known. In considering the unvariable cross-section along the axis of a given beam we become by consecutive integration and with necessary rearrangement the following system of equations:

$$\begin{aligned}
w\,(x) &= 1\,.\,w\,(0) - x\,.\,\varphi\,(0) + \frac{x^2}{2\,E\,J}\,M\,(0) + \frac{x^3}{6\,E\,J}\,Q\,(0) + w_o\,(x)\\
\varphi\,(x) &= 1\,.\,\varphi\,(0) - \frac{x}{E\,J}\,M\,(0) - \frac{x^2}{2\,E\,J}\,Q\,(0) + \varphi_o\,(x)\\
M\,(x) &= 1\,.\,M\,(0) + x\,.\,Q\,(0) + M_o\,(x)\\
Q\,(x) &= 1\,.\,Q\,(0) + Q_o\,(x)\\
q\,(x) &= 1\,.\,q\,(x)
\end{aligned} \tag{8}$$

At the and of the field of the given beam, we can calculate the static quantities with equations (8) by substituting s for x.

The elements of static quantities $w_o\,(x)$, $\varphi_o\,(x)$, $M_o\,(x)$ and $Q_o\,(x)$ written by equations (8) are called loading elements. Their quantities can be calculated with respect to the mode of loading.

b) For a known loading $p\,(x)$, by consecutive integration of equation (7) in any point (x) of the beam we become static quantities by equations:

$$\begin{aligned}
u\,(x) &= 1\,.\,u\,(0) - \frac{x}{E\,A}\,N\,(0) + u_o\,(x)\\
N\,(x) &= 1\,.\,N\,(0) + N_o\,(x)\\
p\,(x) &= 1\,.\,p\,(x)
\end{aligned} \tag{9}$$

3.2.2. *Calculation of loading elements*

In the equations (8) and (9) the loading elements are calculated for single modes of loading of the field of the beam. In practice we find most often the loadings shown in Fig. 3. From loading

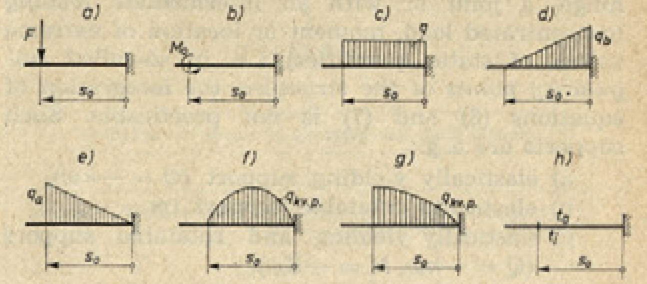


Fig. 3

examples represented (Fig. 3) it is evident that in principle we should calculate for each mode of loading the corresponding loading elements. In this form the calculation is given also in the available literature. Nevertheless, in order to obtain a smaller number of different formulae for loading elements, we shall select the way of simplification, or combination of single loadings. This is very convenient for the digital computer in which, during calculation, we must store many things. Moreover the description of the problem is more simple for typing the data. The calculating accuracy thereby practically is not affected.

With regard to the statement mentioned we shall determine the loading elements or their general expressions for the following modes of loading:

— concentrated loads (Fig. 3a),
— couple of forces (Fig. 3b),
— trapezoidal loading (thereby Fig. 3c, d, e, f, g are comprised),
— temperature loading (Fig. 3h).

Due to differing units used for single elements written in in the matrices and to their multiplication it is highly convenient to arrange the expressions in the nondimensional form. This can be obtained by introducing the so called *comparative quantities* for EJ_c, F_c and s_c. Thus, the static quantities have the following expressions:

$$\begin{aligned}
w_o^* &= w_o\,\frac{E\,J_c}{F_c\,.\,s_c^3}, & Q_o^* &= \frac{Q_o}{F_c}\\
\varphi_o^* &= \varphi_o\,\frac{E\,J_c}{F_c\,.\,s_c^2}, & M_o^* &= \frac{M_o}{F_c\,.\,s_c}\\
u_o^* &= u_o\,\frac{E\,J_c}{F_c\,.\,s_c^3}, & N_o^* &= \frac{N_o}{F_c}\\
q_o^* &= q_o\,\frac{s_c}{F_c}, & p_o^* &= p_o\,\frac{s_c}{F_c}
\end{aligned} \tag{10}$$

3.3. Discontinuities of quantities of state

In the former chapter we have explained how to calculate static quantities of state in single fields for given toadings. At transition of the beam from one field to another field over a support, a hinge, a joint or with an intermediate loading (concentrated load, moment or location of extreme values of static quantities) i. e. in so-called *singularity points* of the structure, the integration of equations (6) and (7) is not practicable. Such supports are e. g.:

a) elastically yielding support ($Q = -kw$),
b) elastically rotatable support ($m = -K\varphi$),
c) elastically yielding and rotatable support ($Q = -kw, M = -K\varphi$),
d) non yielding support in vertical direction ($w = 0$), Q^s,
e) hinge ($M = 0$), φ^s,
f) hinge with zero shearing ($Q = 0$), w^s,
g) vertically guided bearing ($\varphi = 0$), M^s,
h) non yielding support in horizontal direction ($u = 0$),
i) yielding support in horizontal direction ($u \neq 0$) ($N = -k_n u$).

For singularity points, the ascertainment is very important that for any singularity quantities of state we can write so much *additional* conditions (respectively equations) how much *new* unknowns there appear for Q^s, φ^s, w^s and M^s.

3.4. Static state of support "k"

When axial quantities N and u are not considered, the static state of the support "k" is given by four quantities as it is shown in Fig. 4.

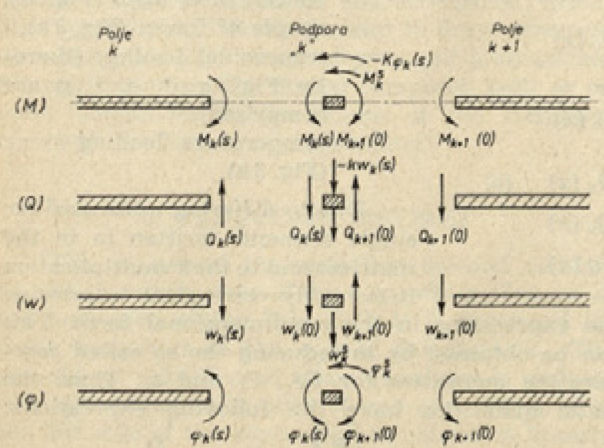


Fig. 4
polje = field, podpora = support

Equilibrium conditions give the connection among the beginning of the field $k + 1$, the support "k" and the end of the field k in the following forms:

$$w_{k+1}(0) = w_k(s) + w_k^s$$
$$\varphi_{k+1}(0) = \varphi_k(s) + \varphi_k^s \qquad (11)$$
$$M_{k+1}(0) = M_k(s) - K\varphi_k(s) + M_k^s \qquad (11)$$
$$Q_{k+1}(0) = Q_k(s) - k\,w_k(s) + Q_k^s$$

In considering the influence of axial quantities N and u we become:

$$u_{k+1}(0) = u_k(s)$$
$$N_{k+1}(0) = N_k(s) - k_n u \qquad (12)$$

3.5. Continuous beams

With these beams we do not consider the axial quantities N and u, because they can be determined directly, when influencing the beam.

Classification of continuous beams:

a) beams with stiff supports ($w = 0$ or $w = $ const.),
b) beams with elastic supports ($w \neq 0$),
c) beams with stiff and elastic supports.

In the calculation of static quantities w, φ, M, Q we have given the classification of girders comprised in one sole *programme* P_k. Therefore, we shall observe in a general form how such girders are solved.

3.5.1. *Initial, boundary and intermediate conditions*

Each end of a beam comprises four static quantities w, φ, M, Q. From Fig. 5 it is evident that on the *beginning*, i. e. on the left-hand support there are always two known static quantities and two unknown quantities. Therefrom it follows

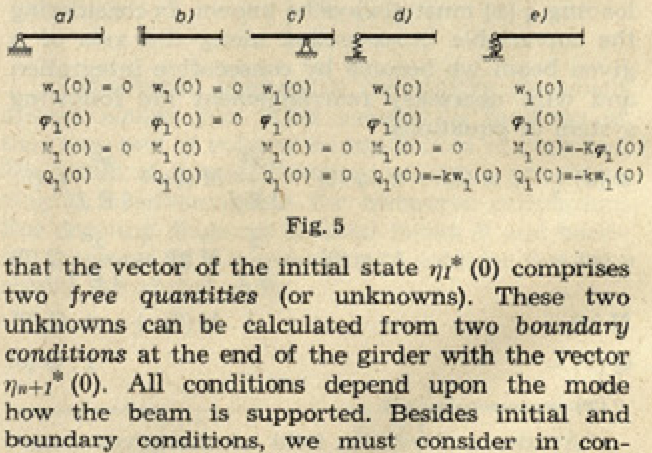


Fig. 5

that the vector of the initial state $\eta_I^*(0)$ comprises two *free quantities* (or unknowns). These two unknowns can be calculated from two *boundary conditions* at the end of the girder with the vector $\eta_{n+I}^*(0)$. All conditions depend upon the mode how the beam is supported. Besides initial and boundary conditions, we must consider in continuous beams also intermediate conditions on supports which appear in point matrices U_i as *discontinuities of quantities of state*, i. e. w^s, φ^s, M^s, Q^s. These are unknowns. In chapter 3.3. we have explained that for any discontinuity quantity we can write a condition (the conjugate quantity s of the discontinuity quantity equals zero) so we obtain such a number of additional equations as there are discontinuity quantities of state.

In treating continuous beams we have in general N unknowns

$$2 + s = N \qquad (13)$$

which can be solved by E equations needed:

$$2 + s = E \qquad (14)$$

In equations (13) and (14) s designates the number of discontinuity quantities.

The initial or left-hand supports of beams are shown in Fig. 5.

3.5.2. *General form of matrix equation for beams with stiff and elastic supports*

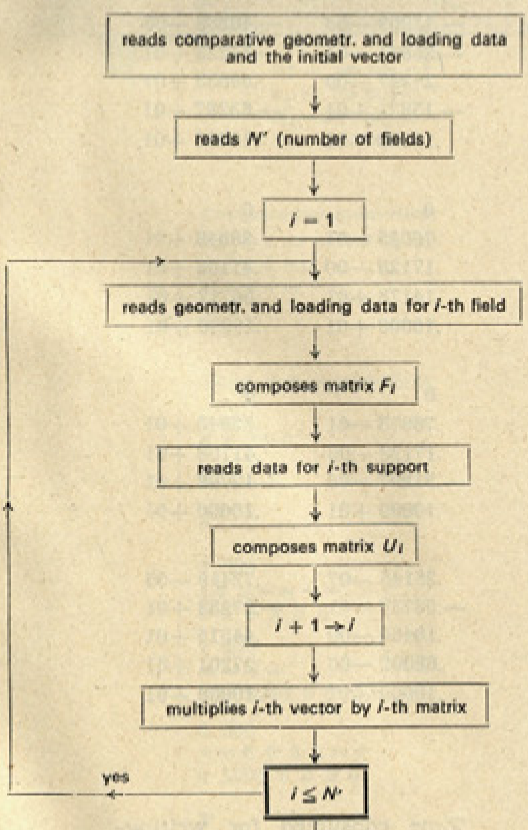
With any number of fields the reduction equation e. g. for the beginning of the field $n+1$ has the following form:

$$\eta_{n+1}^{*}(0) = U_n^{*} . F_n^{*} . U_{n-1}^{*} . F_{n-1}^{*} \ldots \ldots U_1^{*} . F_1^{*} . \eta_1^{*}(0) \qquad (15)$$

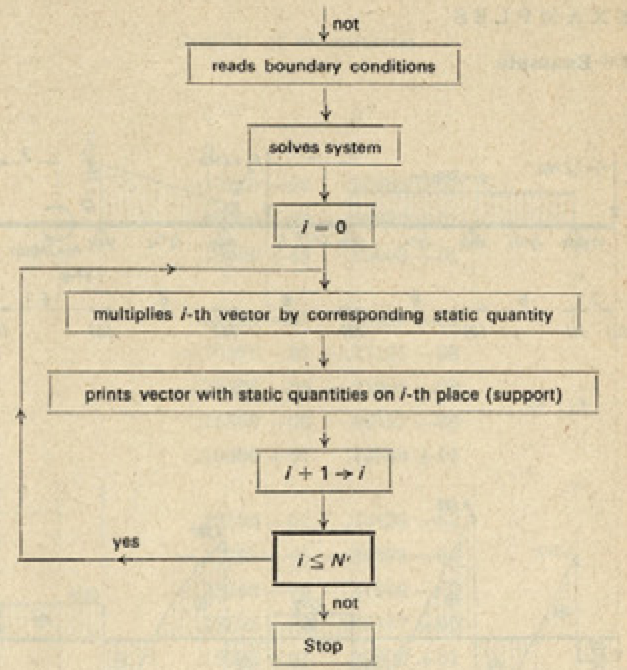
By neglecting axial quantities N and u we can write this equation explicitely in the non-dimensional form.

The explicite form of the matrix equation is written for practical reasons with the elements of the matrix given in non-dimensional form. The calculating programme P_k is so arranged that the computer makes the multiplication of matrices in the non-dimensional form and calculates true values of the results by in advance chosen comparative quantities not before the end after having solved the system of linear equations by Gauss's method. Particular specialities of supports of continuous beams are taken into account in the programme elaborated.

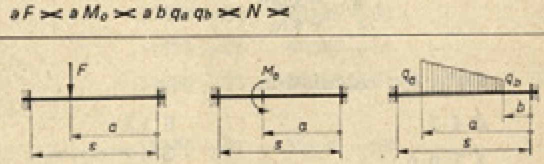
3.5.3. *Flow diagram*



(continuation in the right column)



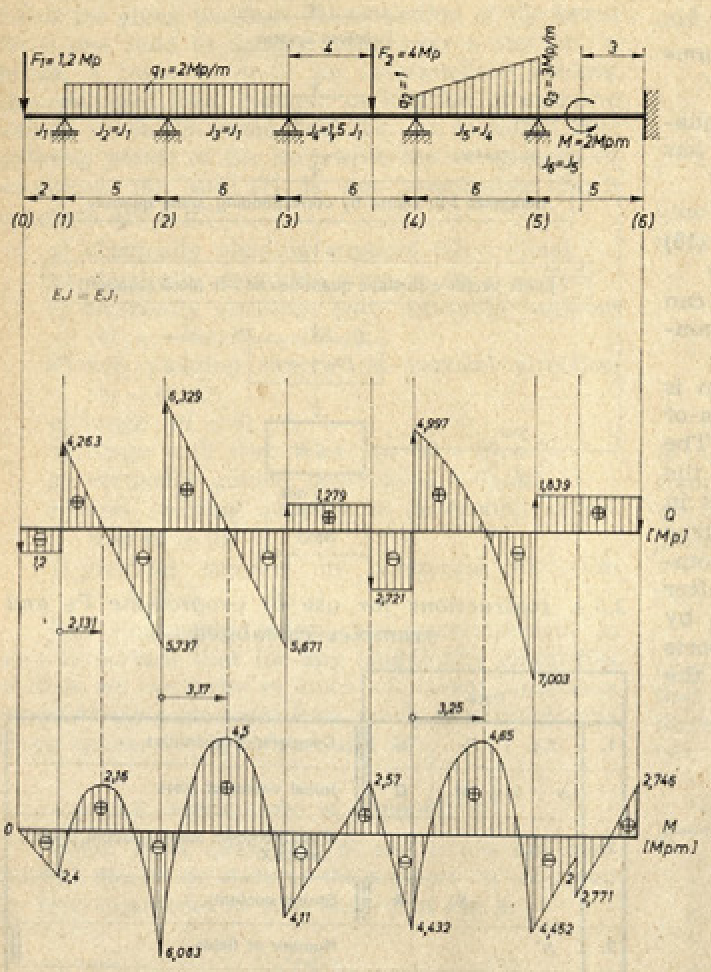
3.5.4. *Instructions for use of programme P_k and examples calculated*

	Data	
1.	EJ_c | F_c | s_c	Comparative quantities
2.	w | φ | M | Q	Initial vector of state
		If am unknown exists (discont. magn.) write X
	$-K$ | $-k$	Spring costants
3.	N'	Number of fields
4.	Field matrix "F_i"	
4a.	EJ_i | s_i	Stiffness and lenght of field
4b.	$a\,F$ >< $a\,M_0$ >< $a\,b\,q_a\,q_b$ >< N ><	
		
5.	Discontinucty (transfer) matrix "U_i"	
5a.	k | K | w^s | φ^s | M^s | Q^s	Discont. quan. If unknown write X
6.	Boundary conditions	
6a.	p'	Number of boundary conditions
6b.	k_i' | l_i' | x_i	Beginning of field (k'_i), stat. quan. (l'_i), size (x_i)
	w 1' | φ 2' | M 3' | Q 4'	Static quantities

Note. It is evident from this table, that only those data should be writen where the line ends with three vertical lines.

EXAMPLES

1st Example



Comparative quantities:

EJ_c = 3 EJ Mpm²
F_c = 4 Mp
s_c = 6 m

Data for calculation

6 4 3

X X 0 0

2'

2 1

2 1,2 ⋈ ⋈ ⋈ ⋈
0 0 0 0 0 X

5 1

⋈ ⋈ 5 0 2 2 ⋈ ⋈
0 0 0 0 0 X

6 1

⋈ ⋈ 6 0 2 2 ⋈

0 0 0 0 0 X

6 1,5
2 4 ⋈ ⋈ ⋈ ⋈
0 0 0 0 0 X
6 1,5
⋈ ⋈ 6 0 1 3 ⋈ ⋈
0 0 0 0 X X
5 1,5
⋈ 3 2 ⋈ ⋈ ⋈
0 0 0 0 0 X
9'
1' 3' 0
1' 4' 0
2' 1' 0
3' 1' 0
4' 1' 0
5' 1' 0
6' 1' 0
7' 1' 0
7' 2' 0

Results of calculation

Reduced res.	True res.	*Stat. quan.*
.17557 —02	.50565 —00	w
.21933 —01	.10528 +01	φ
0	0	M
0	0	Q
.10000 +01	.10000 +01	1

.17462 —09	.50291 —07
—.28066 —01	—.13471 +01
.10000 —00	.23999 +01
.30000 —00	.12000 +01
.10000 +01	.10000 +01

.17462 —09	.50291 —07
—.28066 —01	—.13471 +01
.10000 —00	.23999 +01
—.10658 +01	—.42633 +01
.10000 +01	.10000 +01

—.13969 —08	—.40233 —06
—.35881 —01	—.17223 +01
.25347 —00	.60833 +01
.14341 +01	.57366 +01
.10000 +01	.10000 +01

—.13969 —08	—.40233 —06
—.35881 —01	—.17223 +01
.25347 —00	.60833 +01
—.15821 +01	—.63287 +01
.10000 +01	.10000 +01

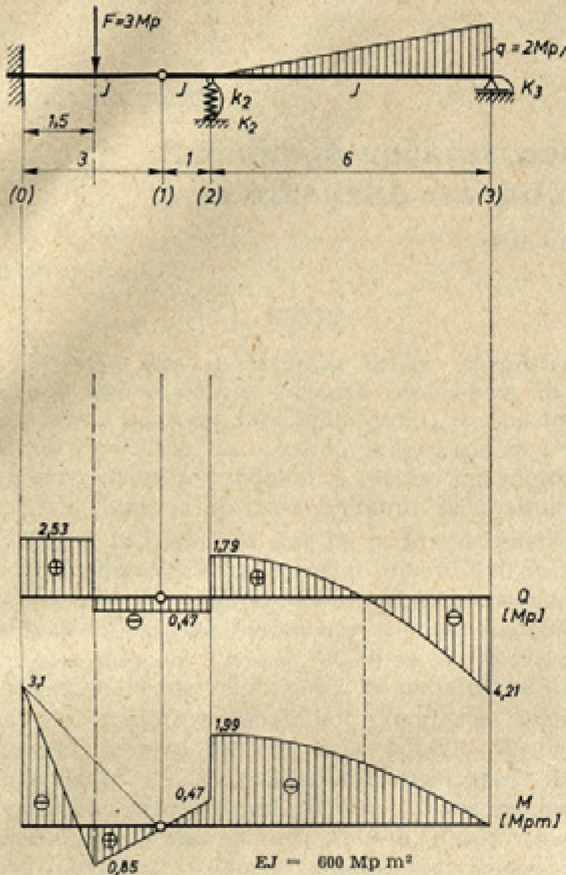
0	0
.76975 —01	.36948 +01
.17128 —00	.41108 +01
.14178 +01	.56712 +01
.10000 +01	.10000 +01

0	0
.76975 —01	.36948 +01
.17128 —00	.41108 +01
—.31997 —00	—.12798 +01
.10000 +01	.10000 +01

.25145 —07	.72419 —05
—.56737 —01	—.27233 +01
.18464 —00	.44315 +01
.68002 —00	.27201 +01
.10000 +01	.10000 +01

Time consumed for writing-in and calculation about 9 minutes.

2nd Example



$EJ = 600$ Mp m²
$k_2 = 300$ Mp/m
$K_2 = 2000$ Mp m
$K_3 = 3000$ Mp m

Comparative quantities:
$EJ_c = 6\ EJ = 3600$ Mp m²
$F_c = 3$ Mp
$s_c = 6$ m

Data for calculation

6 3 3600
0 0 X X

3'

3 600
1,5 3 ⋈ ⋈ ⋈ ⋈
0 0 0 X 0 0

1 600
⋈ ⋈ ⋈ ⋈
300 2000 0 0 0 0

6 600
⋈ ⋈ 6 0 0 2 ⋈ ⋈
0 1000 0 0 0 0

3'
2' 3' 0
4' 1' 0
4' 3' 0

Results of calculation

0	0
0	0
.17200 —00	.30960 +01
—.84400 —00	—.25320 +01
.10000 +01	.10000 +01
.39126 —01	.70428 —02
—.70505 —01	—.21151 —02
—.23283 —09	—.41909 —08
.15599 —00	.46797 —00
.10000 +01	.10000 +01
.39126 —01	.70428 —02
—.12299 —01	—.36899 —03
—.23283 —09	—.41909 —08
.15599 —00	.46797 —00
.10000 +01	.10000 +01
.41898 —01	.75418 —02
—.25299 —01	—.75897 —03
.25998 —01	.46797 —00
.15599 —00	.46797 —00
.10000 +01	.10000 +01
.41898 —01	.75418 —02
—.25299 —01	—.75897 —03
.11032 —00	.19859 +01
—.59818 —00	—.17945 +01
.10000 +01	.10000 +01
—.18160 —07	—.32689 —08
.10728 —00	.32185 —02
.17880 —00	.32185 +01
.14018 +01	.42054 +01
.10000 +01	.10000 +01
—.18160 —07	—.32689 —08
.10728 —00	.32185 —02
—.37252 —08	—.67055 —07
.14018 +01	.42054 +01
.10000 +01	.10000 +01

Time consumed for writing-in and calculation about 6 ½ minutes.

Translated by Branko Vajda

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