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# Uporaba programa MATHEMATICA v metodi končnih elementov za izračun tankih plošč

# The Use of Program MATHEMATICA in the Finite Element Method for Thin Plate Analysis

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*Prispevek prikazuje uporabo programa MATHEMATICA v metodi končnih elementov. S programom smo izpeljali celoten izračun togostne matrike in obtežnega vektorja elementa po metodi deformacije za tanke plošče. Element ima 9 vozlišč s štirimi prostostnimi stopnjami v vozlišču, torej skupaj 36 prostostnih stopenj. S tako izbiro prostostnih stopenj smo dosegli zveznost $C_1$. Ker v uporabljeni literaturi zanj nismo našli označbe, smo mu dodelili označbo H9. Element smo testirali na dveh numeričnih primerih in dobljene rezultate primerjali s točnimi vrednostmi, znanimi iz literature, kakor tudi z rezultati, dobljenimi z izračuni z elementi z manjšim številom prostostnih stopenj (element BFS).*

*The article presents the use of the program MATHEMATICA in the finite element method. The entire computation of the stiffness matrix and load vector was computed with the program. Such an element has 9 node points with four degrees of freedom per node, that is 36 degrees of freedom. Such a choice of degrees of freedom achieves $C_1$ continuity. Because a name for such an element was not found in the literature, it was named H9. The element was tested with numerical examples and the obtained result were compared with exact values and with result obtained by computation with elements with lower value degrees of freedom (element BFS).*

## 0. PREDSTAVITEV PROGRAMA

Program MATHEMATICA je programski paket, ki je zlasti namenjen za uporabo v matematiki, čeprav ga je mogoče koristno uporabiti v mnogih inženirskih problemih, saj ponuja uporabniku široko paleto možnosti. Odlikuje ga zmožnost simboličnega operiranja s podatki. Mnoge naloge je namreč mogoče tako reševati analitično in ne samo numerično.

## 1. PREDSTAVITEV PROBLEMA

Metoda končnih elementov je dandanes ena izmed najbolj razširjenih in uporabnih numeričnih metod, ki se uporabljajo v konstrukterski inženirski praksi. Konstrukcijo, ki jo želimo preračunati, diskretiziramo v končno število elementov in za vsakega izmed njih izračunamo lokalno togostno matriko. Prav računanje togostne matrike elementa je ena od najzahtevnejših operacij, ki porabi največ časa za računanje, še posebej če matriko vsakega elementa numerično integriramo. Kadar se odločimo, da bomo za diskretizacijo konstrukcije uporabili enake vrste elementov (ni nujno tudi enakih velikosti), je primerno takšno matriko ovrednotiti simbolično z integriranjem po ploskvi oz. prostornini elementa.

Ta operacija integriranja pa ni toliko zapletena kakor obsežna naloga, saj pride pri izračunu matrik z večjim številom prostostnih stopenj do množice podatkov, ki vodijo v nepreglednost. Upoštevajoč še človeško nagnjenost k zmotam je sklep ta, da je treba tak izračun prepustiti natančnejšemu izvajalcu - računalniku. Zaradi zmožnosti analitičnega odvajanja, integriranja ter drugih pomembnih možnosti, smo se odločili, da s programom MATHEMATICA izračunamo togostno matriko pravokotnega elementa. Namen tega izračuna ni v prvi vrsti pridobitev togostne matrike, temveč prikaz možnosti uporabe v konstrukterskem področju.

## 0. AN INTRODUCTION OF THE PROGRAM

Program Mathematica is a software package which is intended in the first place for use in mathematics, although its capabilities could also be applied to enginering problems because of its wide specter of functions. One of its best points is its capability of symbolic handling of data. Many tasks can thus be solved in a strictly symbolic way and not only numerically.

## 1. AN INTRODUCTION OF THE PROBLEM

The finite element method is today one of the most widespread and useful methods in engineering practice. A structure which is to be analyzed is discretised into a finite number of elements and the local stiffness matrix must be calculated for each of them. The calculation of the stiffness matrix of the element is one of the most exacting operations and requires most of the computing time, especially if the matrix for each element is integrated numerically. If a decision is made to use the equal type of the element for the discretisation of the structure, it is suitable to valuate such a stiffness matrix in symbolic form with integration over the area or the volume in the element.

This operation of integration is a huge task. Because of its capacity for analytical differentiation, integration and many other important possibilities it was decided to compute stiffness matrix for a rectangle element with the program Mathematica. The main purpose of this computation was not to obtain the stiffness matrix but to demonstrate the possibilities of this program in the structural domain.

## 2. TEORETIČNE OSNOVE

Osnovne enačbe, ki so znane iz mehanike, so privzete iz [1].

Na element je vezan pravokoten kartezijev koordinatni sistem, katerega osi sta vzporedni z robovi plošče. Izhajamo iz izhodiščne predpostavke, da linijski element, ki je bil pred deformacijo pravokoten na ravnino plošče, ostane po deformaciji nespremenjen po dolžini in pravokoten na srednjo deformirano ravnino plošče (Kirchhoff-ova predpostavka). Ta teorija, postavljena že leta 1850, opisuje obnašanje plošče samo z upogibom osrednje ravnine. Upogib le-te je funkcija dveh med seboj pravokotnih koordinat v ravnini plošče. Glede na to predpostavko je naša analiza v dvodimenzionalnem področju. Zapišemo lahko:

## 2. THEORETICAL BACKGROUND

Basic equations from mechanics are taken from [1].

A Cartesian coordinate system is attached to the element, with axes parallel to the boundaries of the plate. We start from the assumption that there is no deformation in the middle plane of the plate and that points lying initially on a normal to middle plane of the plate remain on the normal to middle surface of the plate after deformation (Kirchoff's hypothesis). This classical theory established in 1850 describes the behavior of plates only with the deflection of the middle surface of the plate. This deflection is a function of two orthonormal coordinates in the middle surface of the plate. According to this assumption, our analysis lies in a two dimensional area. We can write:

$$w = w(x, y), \quad u = -z\frac{\partial w}{\partial x} \quad \text{in} \quad v = -z\frac{\partial w}{\partial y} \qquad (1).$$

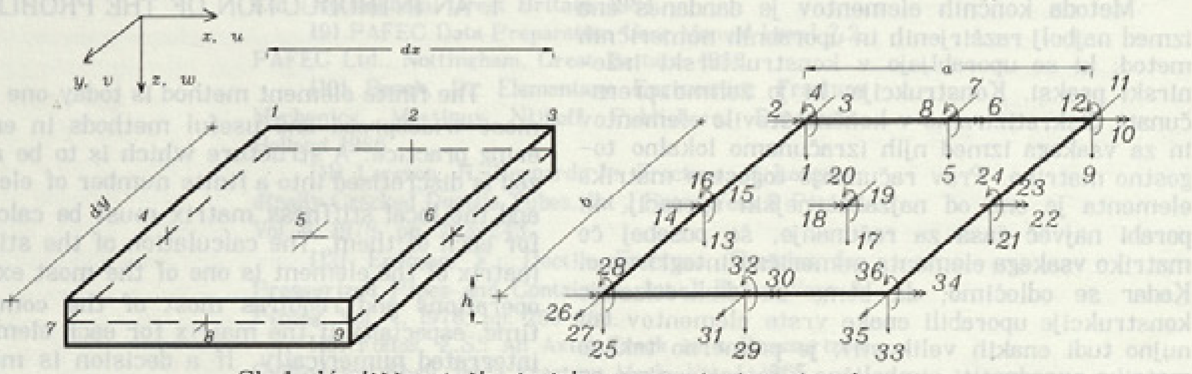
Komponente deformacijskega tenzorja so:

The components of the strain tensor are:

$$\varepsilon = \begin{bmatrix} -z\dfrac{\partial^2 w}{\partial x^2} & -2z\dfrac{\partial^2 w}{\partial x \partial y} & 0 \\ -2z\dfrac{\partial^2 w}{\partial x \partial y} & -z\dfrac{\partial^2 w}{\partial y^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (2).$$

Končni element je po tlorisni obliki pravokotnik. Število vozliščnih točk na elementu je devet, njihova lokacija je naslednja: štiri vogalne točke, štiri točke na sredini stranic pravokotnika in sredinska točka (sl. 1).

The shape of the finite element is rectangle. The number of nodes is 9. They are located as follows: four corner points, four points on the midside nodes and the centroidal point (Fig. 1).



Sl. 1. *Vozliščne točke in izbrane prostostne stopnje.*

Fig. 1. *The nodal points and chosen degrees of freedom.*

Zaradi zagotovitve čim večje natančnosti za glavne neznanke v vozliščih izberemo pomik $w$, prva odvoda $\partial w/\partial x$ in $\partial w/\partial y$, (ki sta sorazmerna zasukoma $\Theta_x$ in $\Theta_y$) ter mešani odvod $\partial^2 w/\partial x \partial y$; torej imamo v vozlišču 4 prostostne stopnje oziroma 36 prostostnih stopenj v celotnem elementu. Vektor neznak v vozlišču »i« zapišemo kot:

To achieve better accuracy, four unknowns in the nodes are next chosen: displacements $w$, first derivates $\partial w/\partial x$ and $\partial w/\partial y$ (which are proportional to the rotations $\Theta_x$ in $\Theta_y$) and mixed derivative $\partial^2 w/\partial x \partial y$. According to chosen unknowns we have 4 degrees of freedom per node or 36 degrees of freedom per total element. The vector of the unknowns in the joint can be written as:

$$q_i = \begin{Bmatrix} w \\ a\Theta_x \\ b\Theta_y \\ \partial^2 w/\partial x \partial y \end{Bmatrix} \qquad i = 1, 2, 3, \ldots, 9 \qquad (3).$$

Vrednosti $a\,\Theta_x$ in $b\,\Theta_y$ sta uvedeni zato, da postane vektor neznanih količin dimenzionalno homogen.

Za aproksimacijo polja pomikov izberemo interpolacijske funkcije v obliki Hermitovih polinomov. Izbira Hermitovih (podobno kakor Lagrangeovih) polinomov ima prednost, da omogoča v večdimenzionalnih problemih preprost izračun iskane interpolacijske funkcije s produktom enodimenzionalnih interpolacijskih funkcij. Prikazali bomo izračun Hermitovih polinomov v eni dimenziji. Na elementu dolžine »*l*«, vpeljemo naravne koordinate s substitucijo $\xi = x/l$. Poiskati moramo šest polinomov $H_i\ (x)$, ki bodo zadovoljevali enačbo:

The values $a\,\Theta_x$ are $b\,\Theta_y$ are introduced to make the vector of unknowns dimensionally homogeneous.

For the approximation of the field of the displacements, Hermitian polynomials are chosen as interpolating functions. The choice of Hermite polynomials (as with Lagrangean polynomials) has the advantage of enabling simple evaluation of several dimensional interpolating functions as the product of one dimensional interpolating function. We will show the evaluation of Hermite polynomials in one dimension on an element with length »*l*«. We will introduce the natural coordinate $\xi = x/l$. We must find six polynomials $H_i\,(x)$ to fit (satisfy) the equation:

$$w(\xi) = H_1(\xi)\, w(0) + H_2(\xi)\, \frac{d\,w\,(0)}{d\,\xi} + H_3(\xi)\, w\,(0{,}5) + H_4(\xi)\, \frac{d\,w\,(0{,}5)}{d\,\xi} + H_5(\xi)\, w\,(1) + H_6(\xi)\, \frac{d\,w\,(1)}{d\,\xi} \tag{4}$$

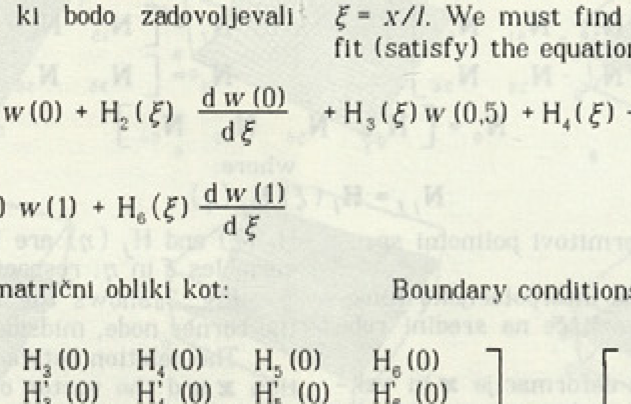
Robne pogoje zapišemo v matrični obliki kot:

Boundary conditions are in matrix form:

$$\begin{bmatrix} H_1(0) & H_2(0) & H_3(0) & H_4(0) & H_5(0) & H_6(0) \\ H_1'(0) & H_2'(0) & H_3'(0) & H_4'(0) & H_5'(0) & H_6'(0) \\ H_1(1/2) & H_2(1/2) & H_3(1/2) & H_4(1/2) & H_5(1/2) & H_6(1/2) \\ H_1'(1/2) & H_2'(1/2) & H_3'(1/2) & H_4'(1/2) & H_5'(1/2) & H_6'(1/2) \\ H_1(1) & H_2(1) & H_3(1) & H_4(1) & H_5(1) & H_6(1) \\ H_1'(1) & H_2'(1) & H_3'(1) & H_4'(1) & H_5'(1) & H_6'(1) \end{bmatrix} = \begin{bmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{bmatrix}$$

Hermitovi polinomi v naravnih koordinatah dobijo obliko (sl. 2):

Hermitian polynomials in the natural coordinates are (Fig. 2)




Sl. 2. *Hermitovi polinomi prvega reda.*
Fig. 2. *The Hermite 's polynomials of first degree.*

$$\begin{aligned} H_1\ (\xi) &= 1 - 23\,\xi^2 + 66\,\xi^3 - 68\,\xi^4 + 24\,\xi^5 \\ H_2\ (\xi) &= \xi - 6\,\xi^2 + 13\,\xi^3 - 12\,\xi^4 + 4\,\xi^5 \\ H_3\ (\xi) &= 16\,\xi^2 - 32\,\xi^3 + 16\,\xi^4 \\ H_4\ (\xi) &= -8\,\xi^2 + 32\,\xi^3 - 40\,\xi^4 + 16\,\xi^5 \\ H_5\ (\xi) &= 7\,\xi^2 - 34\,\xi^3 + 52\,\xi^4 - 24\,\xi^5 \\ H_6\ (\xi) &= -\xi^2 + 5\,\xi^3 - 8\,\xi^4 + 4\,\xi^5 \end{aligned}$$

Polje pomikov na našem (dvodimenzionalnem) elementu definiramo kot:

The field of displacement on our two dimensional element is defined as:

$$w = w(\xi, \eta) = \mathbf{N}(\xi, \eta)\, \boldsymbol{q} \qquad (5),$$

kjer velja:
$\eta = y/l$, naravna koordinata v smeri osi $y$,
$\boldsymbol{q} = [\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \ldots \boldsymbol{q}_9]^T$, vektor vozliščnih pomikov oziroma neznank (3),
$\mathbf{N}(\xi, \eta)$, matrika interpolacijskih funkcij v obliki:

where:
$\eta = y/l$, the natural coordinate in $y$ direction,
$\boldsymbol{q} = [\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \ldots \boldsymbol{q}_9]^T$, vector of modal displacements (3),
$\mathbf{N}((\xi, \eta)$, the matrix of the interpolating functions:

$$\mathbf{N}(\xi, \eta) = [\mathbf{N}_1, \mathbf{N}_2, \ldots, \mathbf{N}_9] \qquad (6).$$

Vsaka izmed podmatrik $\mathbf{N}_i$ (i = 1, 2, ..., 9) ustreza enemu vozlišču in ima štiri elemente:

Each of the sub matrixes $\mathbf{N}_i$ (i = 1, 2, ..., 9) belongs to one node and it has four elements:

$$\mathbf{N}_1 = [\mathbf{N}_{11}\ \ \mathbf{N}_{12}\ \ \mathbf{N}_{21}\ \ \mathbf{N}_{22}] \qquad \mathbf{N}_5 = [\mathbf{N}_{33}\ \ \mathbf{N}_{34}\ \ \mathbf{N}_{43}\ \ \mathbf{N}_{44}]$$
$$\mathbf{N}_2 = [\mathbf{N}_{31}\ \ \mathbf{N}_{32}\ \ \mathbf{N}_{41}\ \ \mathbf{N}_{42}] \qquad \mathbf{N}_6 = [\mathbf{N}_{53}\ \ \mathbf{N}_{54}\ \ \mathbf{N}_{63}\ \ \mathbf{N}_{64}]$$
$$\mathbf{N}_3 = [\mathbf{N}_{51}\ \ \mathbf{N}_{52}\ \ \mathbf{N}_{61}\ \ \mathbf{N}_{62}] \qquad \mathbf{N}_7 = [\mathbf{N}_{15}\ \ \mathbf{N}_{16}\ \ \mathbf{N}_{25}\ \ \mathbf{N}_{26}]$$
$$\mathbf{N}_4 = [\mathbf{N}_{13}\ \ \mathbf{N}_{14}\ \ \mathbf{N}_{23}\ \ \mathbf{N}_{24}] \qquad \mathbf{N}_8 = [\mathbf{N}_{35}\ \ \mathbf{N}_{36}\ \ \mathbf{N}_{45}\ \ \mathbf{N}_{46}]$$
$$\mathbf{N}_9 = [\mathbf{N}_{55}\ \ \mathbf{N}_{56}\ \ \mathbf{N}_{65}\ \ \mathbf{N}_{66}] \qquad (7),$$

kjer velja:

where:

$$\mathbf{N}_{ij} = \mathrm{H}_i(\xi)\, \mathrm{H}_j(\eta) \qquad (8)$$

in so $\mathrm{H}_i(\xi)$ in $\mathrm{H}_j(\eta)$ Hermitovi polinomi spremenljivk $\xi$ in $\eta$.

$\mathrm{H}_i(\xi)$ and $\mathrm{H}_j(\eta)$ are Hermitian polynomials of variables $\xi$ in $\eta$, respectively.

Na sliki 3 so prikazane interpolacijske funkcije za vogalno vozlišče, vozlišče na sredini roba in osrednje vozlišče.

Fig. 3 shows the interpolating function for the corner node, midside node and central node.

Zveza med vektorjem deformacije $\boldsymbol{\varkappa}$ in vektorjem osnovnih neznanih količin v vozlišču je podana z izrazom:

The relation between the vector of deformation $\boldsymbol{\varkappa}$ and the vector of the unknows in joint is given by the expression:

$$\boldsymbol{\varkappa} = -\begin{bmatrix} \dfrac{1}{a^2}\dfrac{\partial^2 w}{\partial \xi^2} \\[2ex] \dfrac{1}{b^2}\dfrac{\partial^2 w}{\partial \eta^2} \\[2ex] \dfrac{2}{ab}\dfrac{\partial^2 w}{\partial \xi\, \partial \eta} \end{bmatrix} \qquad (9).$$

ki ob vpeljavi (5) in (6) postane:

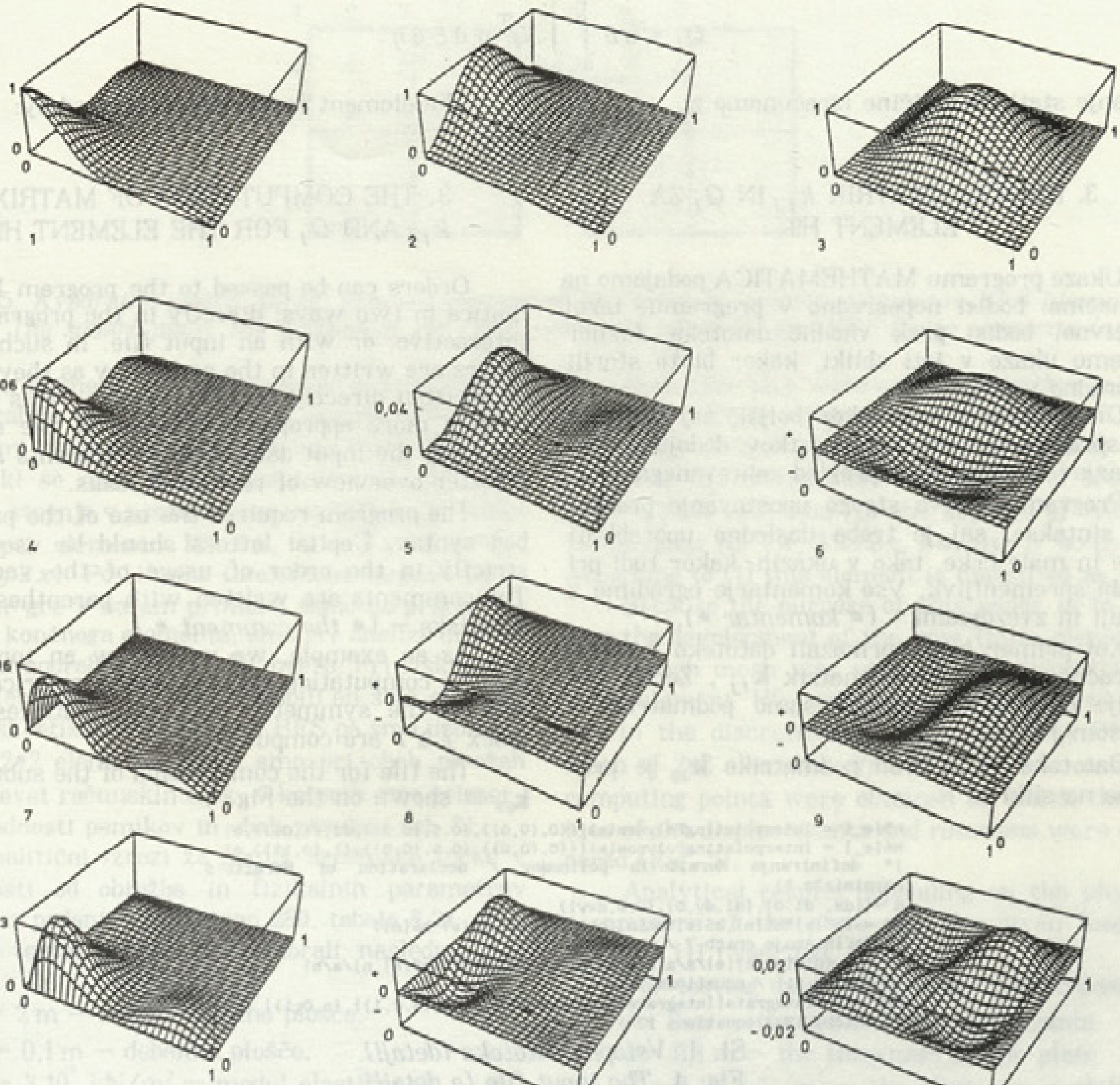
Introducing Eq. (5) and Eq. (6) it follows:

$$\boldsymbol{\varkappa} = -\begin{bmatrix} \dfrac{1}{a^2}\dfrac{\partial^2 \mathrm{N}}{\partial \xi^2} \\[2ex] \dfrac{1}{b^2}\dfrac{\partial^2 \mathrm{N}}{\partial \eta^2} \\[2ex] \dfrac{2}{ab}\dfrac{\partial^2 \mathrm{N}}{\partial \xi\, \partial \eta} \end{bmatrix} \cdot q = B\, q = [B_1\, B_2, \ldots, B_9] \qquad (10),$$

kjer so:

where

$$B_I = -\begin{bmatrix} \dfrac{1}{a^2}\dfrac{\partial^2 \mathrm{N}_I}{\partial \xi^2} \\[2ex] \dfrac{1}{b^2}\dfrac{\partial^2 \mathrm{N}_I}{\partial \eta^2} \\[2ex] \dfrac{2}{ab}\dfrac{\partial^2 \mathrm{N}_I}{\partial \xi\, \partial \eta} \end{bmatrix}, \quad I = 1, 2, \ldots 9 \qquad (11).$$




Sl. 3. *Interpolacijske funkcije vogalne, robne in sredinske točke.*
Fig. 3. *The interpolating functions for the corner node, midside node and the middle point.*

Togostno matriko elementa oblikujemo iz posameznih podmatrik:

The element stiffnes matrix is formed from:

$$K = [k_{ij}], \quad i, j = 1, 2, \ldots 9 \qquad (12).$$

kjer so:

where

$$k_{ij} = ab \int_0^1 \int_0^1 B_i^{\mathrm{T}}(\xi, \eta) D B_j(\xi, \eta) \, \mathrm{d}\xi \, \mathrm{d}\eta \qquad 13)$$

in je D matrika koeficientov elastičnosti:

and D is the matrix of elastic coefficients.

$$D = \frac{E\,h^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \qquad (14).$$

Podobno izračunamo vektor posplošenih sil v vozlišču *i*, ki je funkcija površinske razdelitve obtežbe po elementu:

In a similar manner, the vector of the generalized forces in the joint *i* is obtained. This vector is the function of the distribution of the element surface load as:

$$Q_I = a\,b \int_0^1 \int_0^1 N_I^{\mathrm{T}}\, p \,\mathrm{d}\xi\, \mathrm{d}\eta \tag{15}$$

Notranje statične količine izračunamo z:

The element forces are evaluated by:

$$M = D\,B\,q \tag{16}$$

## 3. IZRAČUN MATRIK $k_{IJ}$ IN $Q_I$ ZA ELEMENT H9

Ukaze programu MATHEMATICA podajamo na dva načina: bodisi neposredno v programu, torej iterativno, bodisi prek vhodne datoteke, kamor zapišemo ukaze v isti obliki, kakor bi to storili neposredno v programu.

Drugi način je vsekakor boljši, saj omogoča lažje spreminjanje vhodnih podatkov, dodajanje komentarjev in celovitejši pregled zahtevanega dela.

Program zahteva strogo upoštevanje predpisane sintakse, saj je treba dosledno uporabljati velike in male črke, tako v ukazih, kakor tudi pri uporabi spremenljivk. Vse komentarje ogradimo z oklepaji in zvezdicami - (* *komentar* *).

Kot primer bomo prikazali datoteko z ukazi za izračun ene izmed podmatrik $\mathbf{k}_{IJ}$. Zaradi simetrije dejansko računamo samo podmatrike z indeksom $J \geqq I$.

Datoteka za izračun podmatrike $\mathbf{k}_{99}$ je prikazana na sliki 4.

## 3. THE COMPUTATION OF MATRIX $k_{IJ}$ AND $Q_I$ FOR THE ELEMENT H9

Orders can be passed to the program Mathematica in two ways: directly in the program, i.e. interactive, or with an input file. In such a file orders are written in the same way as they would be written directly in the program. This second way is more appropriate because of the ease of changing the input data, adding comments and for a better overview of requested tasks.

The program requires the use of the prescribed syntax. Capital letters should be used very strictly in the order of usage of the variables. The comments are written with parentheses and asterisks — (* *the comment* *).

As an example, we will show an input file for the computation of one of the matrices $\mathbf{k}_{IJ}$. Due to the symmetry only the matrices with index $J \geqq I$ are computed.

The file for the computation of the submatrix $\mathbf{k}_{99}$ is shown on the Fig. 4.

```
h5[e_] = InterpolatingPolynomial[{{0,{0,0}},{0.5,{0,0}},{1,{1,0}}},e]
h6[e_] = InterpolatingPolynomial[{{0,{0,0}},{0.5,{0,0}},{1,{0,1}}},e]
(* definiranje Hermitovih polinomov / declaration of Hermite's
poynimials *)
d ={{dx, d1,0},{d1,dy,0},{0,0,dxy}}
n9 = {h5[e]*h5[n],h5[e]*h6[n],h6[e]*h5[n],h6[e]*h6[n]}
(* definiranje enacb 7 / equations 7 *)
b9  = -{D[D[n9,e],e]/a/a,D[D[n9,n],n]/b/b,2*D[D[n9,e],n]/a/b}
(* enacbe 11  / equations 11 *)
k99 =a*b*Integrate[Integrate[Transpose[b9].d.b9,{e,0,1}],{n,0,1}]
(* enacbe 13 / equations 13*)
```

Sl. 4. *Vstopna datoteka (detajl).*
Fig. 4. *The input file (a detail).*

Datoteko z ukazi podamo programu z ukazom *<<imedoteke.tip*.

Podobno izračunamo preostalih 44 matrik $\mathbf{k}_{IJ}$ reda 4*4. Program izvede zadane ukaze, vendar ne poišče avtomatično optimalne (najkrajše) oblike zapisa rezultatov.

Vse rezultate je mogoče shraniti v datoteko; na izbiro imamo več različnih oblik zapisa. Poleg standardnega zapisa so mogoči še zapisi v programskih jezikih C in Fortran ter zapis v obliki, ki jo razume urejevalnik besedila TEX. Možno je oblikovati tudi svoje oblike zapisov.

### 3.1 **Testni primer**

Analiza kvadratne plošče z zvezno obtežbo (sl. 5). Zaradi dvoosne simetrije je dovolj analizirati samo četrtino plošče (na dveh robovih upoštevamo realne robne pogoje, na nasprotnih robovih pa simetrijske robne pogoje = zasuk okoli roba je enak nič).

Uporabnost elementa H9 smo preizkusili na dveh primerih:

a) plošča je na obodu vpeta in

b) plošča je na obodu prosto položena.

The input file is transmitted to the program with a command *«filename.type*.

In the similar way, the remaining 44 matrices $\mathbf{k}_{IJ}$ of order 4*4 are calculated. The program executes the given commands, but does not discover a shorter form of the result automatically.

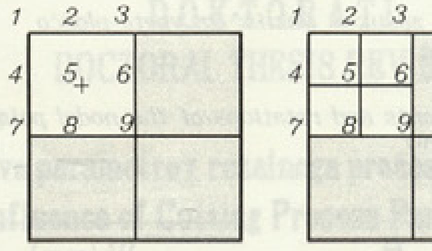
It is possible to store all the obtained results on a file. There are several possibilities of format; in addition to the standard format, there are formats in programming languages such as C and Fortran and a format for the text editor TEX. It is possible to create new formats.

### 3.1 **Test Examples**

The analysis of a rectangular plate with uniformly distributed load (Fig. 5). Because of the double symmetry, only one quarter of the plate was analyzed. On two boundaries, the real boundary conditions were considered, and two of the symmetrical boundary conditions. The usefulness of the element H9 was tested in two examples:

a) a plate with clamped boundaries,

b) a simply supported plate.




Sl. 5. *Kvadratna plošča: mreži končnih elementov H9 (levo) in končnih elementov BFS (desno).*
Fig. 5. *The square plate: the meshes of the finite elements H9 (left) and BFS finite elements (right).*

Oba primera smo preračunali s programom, ki uporablja končni element s šestnajstimi prostostnimi stopnjami. Element ima štiri vozliščne točke, ki se ujemajo z vogalnimi točkami, prostostne stopnje v posameznem vozlišču pa so enake kakor pri elementu H9 ($w$, $\partial w/\partial x$, $\partial w/\partial y$, $\partial^2 w/\partial y \partial x$). Po [1] ima ta element oznaćbo BFS.

Ker gre v našem primeru samo za prikaz izpeljave končnega elementa, smo pri analizi in preverjanju uporabili zelo grobo mrežo. Pri diskretizaciji z elementom H9 smo uporabili en sam element , pri diskretizaciji z elementi BFS pa smo uporabili mrežo 2*2 elementa. Tako smo pri obeh mrežah dobili devet računskih točk, v katerih smo primerjali vrednosti pomikov in obeh zasukov (sl. 5).

Analitični izrazi za pomik sredinske točke v odvisnosti od obtežbe in fizikalnih parametrov plošče so podani v [1] (stran 280, tabela 8.2).

Za testni primer smo izbrali naslednje podatke:

$l$ = 2 m — razpon celotne plošče,
$h$ = 0,1 m — debelina plošče,
$E = 3.10^7$ kN/m$^2$ — modul elastičnosti
$\nu$ = 0,3 — Poissonov koeficient,
$q$ = 10 kN/m$^2$ — zvezna obtežba plošče.

*1. primer*

Plošča je na obodu vpeta. Analitična vrednost pomika v sredini plošče (točka 9) je 0,0733824 mm [1]. Rezultati analize so podani v preglednici 1.

*2. primer*

Plošča je na obodu prosto položena. Analitična vrednost pomika v sredini plošče (točka 9) je 0,2364544 mm [1]. Rezultati analize so podani v preglednici 2.

Kakor je razvidno iz preglednic 1 in 2, je en sam element H9 pri analizi vpete plošče dosegel malenkostno boljše rezultate kakor mreža štirih elementov BFS. Nasprotno pa je element H9 dosegel pri prosto položeni plošči nekoliko slabše rezultate. Šele analiza konstrukcije, diskretizirane z gostejšo mrežo, bi dala pravi odgovor na vprašanje o večji ustreznosti elementa H9 v primerjavi z elementom BFS; vendar to presega obseg tega prispevka.

Both examples were computed with a program that uses a finite element with 16 degrees of freedom. This element has four nodal points that coincide with the corners and the degrees of freedom in each node are equal to the degrees of freedom for element H9 ($w$, $\partial w/\partial x$, $\partial w/\partial y$, $\partial^2 w/\partial y \partial x$). According to [1] this element is classified as BFS.

Because the purpose of this paper is to present the development of the new finite element, a very rough mesh was used. In the discretization with element H9, only one element was used and in the discretization with elements BFS, a mesh of 2*2 elements was used. In both cases, 9 computing points were obtained in which the values of the displacements and rotations were compared (Fig. 5).

Analytical results depending on the physical parameters of the plate and the given load are given in [1] (page 280, table 8.2).

For testing data values were next chosen:

$l$ = 2 m — the span of the whole plate
$h$ = 0,1 m — the thickness of the plate
$E = 3.10^7$ kN/m$^2$ — the elasticity module
$\nu$ = 0,3 — the Poisson' s ratio
$q$ = 10 kN/m$^2$ — the value of the uniform load

*Case 1*

Clamped plate. The exact value of the deflection in the centre of the plate (point 9) is 0.0733824 mm [1]. The results of the analysis are given in table 1.

*Case 2*

Simply supported plate. The exact value of the deflection in the centre of the plate (point 9) is 0.2364544 mm [1]. The results of the analysis are given in table 2.

It is evident from tables 1 and 2 that in the case of a clamped plate, slightly better results were obtained by the element H9. On the other hand, the element H9 produced the worst results in the analysis of the simply supported plate. Only an analysis of the structure, discretised with a finer mesh would answer the question about the superiority of element H9 to element GFS. This exceeds the purpose of this paper.

Preglednica 1: *Primerjava pomikov in zasukov vozlišč za vpeto ploščo*
a) elementi H9 (en element)
b) elementi BFS (mreža 2 * 2)

Table 1: *The comparison of displacements and rotations of the nodal points for the clamped plate*
a) elements H9 (one element)
b) elements BFS (mesh 2 * 2)

| | a) | | | | b) | | |
|---|---|---|---|---|---|---|---|
| | W (mm) | $\Theta_x$ | $\Theta_y$ | | W (mm) | $\Theta_x$ | $\Theta_y$ |
| 1 | 0,000000 | 0,000000 | 0,000000 | 1 | 0,00000 | 0,00000 | 0,00000 |
| 2 | 0,000000 | 0,000000 | 0,000000 | 2 | 0,00000 | 0,00000 | 0,00000 |
| 3 | 0,000000 | 0,000000 | 0,000000 | 3 | 0,00000 | 0,00000 | 0,00000 |
| 4 | 0,000000 | 0,000000 | 0,000000 | 4 | 0,00000 | 0,00000 | 0,00000 |
| 5 | 0,026868 | 0,063568 | 0,063568 | 5 | 0,02684 | 0,06333 | 0,06333 |
| 6 | 0,044213 | 0,106828 | 0,000000 | 6 | 0,04416 | 0,10674 | 0,00000 |
| 7 | 0,000000 | 0,000000 | 0,000000 | 7 | 0,00000 | 0,00000 | 0,00000 |
| 8 | 0,044213 | 0,000000 | 0,106828 | 8 | 0,04416 | 0,00000 | 0,10674 |
| 9 | 0,073688 | 0,000000 | 0,000000 | 9 | 0,07367 | 0,00000 | 0,00000 |

Preglednica 2: *Primerjava pomikov in zasukov vozlišč za prosto položeno ploščo*
a) elementi H9 (en element)
b) elementi BFS (mreža 2 * 2)

Table 2: *The comparison of displacements and rotations of the nodal points for the simply supported plate*
a) elements H9 (one element)
b) elements BFS (mech 2 * 2)

| | a) | | | | b) | | |
|---|---|---|---|---|---|---|---|
| | W (mm) | $\Theta_x$ | $\Theta_y$ | | W (mm) | $\Theta_x$ | $\Theta_y$ |
| 1 | 0,000000 | 0,000000 | 0,000000 | 1 | 0,00000 | 0,00000 | 0,00000 |
| 2 | 0,000000 | 0,271169 | 0,000000 | 2 | 0,00000 | 0,27841 | 0,00000 |
| 3 | 0,000000 | 0,363006 | 0,000000 | 3 | 0,00000 | 0,37594 | 0,00000 |
| 4 | 0,000000 | 0,000000 | 0,271169 | 4 | 0,00000 | 0,00000 | 0,27841 |
| 5 | 0,112005 | 0,162521 | 0,162521 | 5 | 0,11715 | 0,17884 | 0,17884 |
| 6 | 0,152476 | 0,237735 | 0,000000 | 6 | 0,16163 | 0,24774 | 0,00000 |
| 7 | 0,000000 | 0,000000 | 0,363006 | 7 | 0,00000 | 0,00000 | 0,37594 |
| 8 | 0,152476 | 0,000000 | 0,237735 | 8 | 0,16163 | 0,00000 | 0,24774 |
| 9 | 0,212448 | 0,000000 | 0,000000 | 9 | 0,22456 | 0,00000 | 0,00000 |

## 5. SKLEP

Element, katerega izpeljavo smo prikazali, ni novost, zanimiv je samo način izpeljave. Prikazani način omogoča zanesljivo izpeljavo obsežnega končnega elementa v razmeroma kratkem času. Takšen element je z osebnim računalnikom PC 486 namreč mogoče izračunati v enem delovnem dnevu, z zmožnejšimi računalniki pa še znatno hitreje. Izračun končnega elementa s programom MATHEMATICA poleg hitrosti ponuja še točnost izračuna, saj ob pravilno podanih podatkih odpadejo vse človeške zmote pri računu. Zato lahko upravičeno trdimo, da je tak program v konstrukterskem delu več ko dobrodošel, njegova uporabnost pa se bo zagotovo izkazala še pri marsikaterem drugem inženirskem problemu.

## 5. LITERATURA
## 5. REFERENCES

[1] Sekulovič, M.: Metod konačnih elemenata. Gradjevinska knjiga, Beograd, 1988

[2] Wolfram, S.: Mathematica. A System of Doing Mathematics by Computer. The Advance Book Program. 1991.

## 5. CONCLUSION

The element whose derivation was demonstrated in this paper is not an innovation. The point of interest lies in its derivation. The presented way enables a precise derivation of a large finite element in short time. It is possible to develop and compute such an element with a PC486 machine in one day of work. Such an approach offers in addition to speed, also accuracy. The use of the program Mathematica can be welcomed in the field of engineering work and its advantages are likely to apply to many other engineering problems.

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