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Plastičnost vrtečega se nasajenega hiperboličnega diska
Plastic Yield of a Rotating Compound Hyperbolic Disk

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0. UVOD

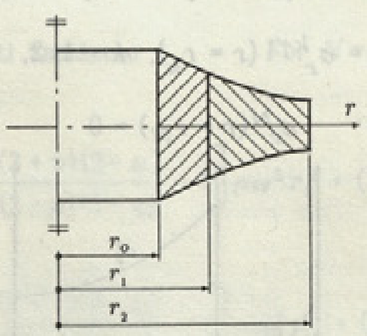
S sestavljanjem sosrednih obročev lahko spremenimo potek radialnih in obročnih napetosti, ki se v njih pojavijo zaradi delovanja obremenitev. Notranji obroč je pod tlačno napetostjo, medtem ko je zunanji obroč pod povečano natezno napetostjo [1].

Obravnavajmo N sosrednih obročev, izdelanih iz enakega gradiva, pri čemer naj ima disk spremenljivo debelino, ki jo privzemimo s hiperboličnim potekom (sl. 1):

0. INTRODUCTION

In compounding of concentric rings, use is made of fact that the radial and circumferential stresses, due to internal pressure, fall off rapidly from the inside surface in the thick wall. The inner ring is put into compression, while the outher ring carries a higher tensile stress that it would otherwise do [1].

In the case considered, a set of N concentric rings, is made of the same material, composing the disk of variable thickness, assuming hyperbolic decrease w.r.t. radii (fig. 1):



Sl. 1. *Sestavljena obroča spremenljive debeline.*
Fig. 1. *Compound rings of variable thickness.*

$$t(r) = t_{k-1}\left(\frac{r_{k-1}}{r}\right)^{n}, \quad r_{k-1} \leq r \leq r_k, \quad k = 1, 2, \ldots, N, \quad n > 0 \tag{1}$$

Obremenitev naj bo z notranjim tlakom in s centrifugalno silo. Temperaturnih sprememb ne bomo upoševali. Z uporabo kriterija Tresca za začetek plastifikacije izračunajmo optimalno porazdelitev zaporednih obročev, če pride do tečenja snovi hkrati na vseh notranjih površinah posameznih obročev [2].

It is subject to internal pressure and centrifugal force. No temperature gradient is taken into account. Using Tresca yield criterion, optimum distribution of consecutive rings is considered, assuming that the yielding occurs simultaneously at the inner surfaces of each individual ring [2].

1. TEORETIČNA OBRAVNAVA

1. THEORY

Zaradi ravnovesne enačbe v vsakem delu:

Due to the equilibrium in each part:

$$\sigma_{\Phi}^{k} = r\frac{d\sigma_r^k}{dr} + (1-n)\,\sigma_r^{k} + \rho\omega^2 r^2 \tag{2}$$

določa elastično obnašanje naslednja diferencialna enačba:

the elastic case is determined by the differential equation:

$$\frac{d^2\sigma_r^k}{dr^2} + \frac{3-n}{r}\frac{d\sigma_r^k}{dr} - \frac{n(1+\nu)}{r^2}\sigma_r^k + (3+\nu)\,\rho\omega^2 = 0 \tag{3},$$

katere rešitev za radialno napetost je podana v obliki:

with its solution:

$$\sigma_r^k = A_k r^p - B_k{}^q - \frac{3+\nu}{(2-p)(2-q)}\,\rho\omega^2 r^2 \tag{4}$$

in sta

where

$$p,\, q = \frac{n}{2} - 1 \pm \sqrt{(n/2)^2 + \nu n + 1} \tag{5}.$$

Obročno napetost dobimo z enačbo (2):

The hoop stress is now from eq. (2):

$$\sigma_{\Phi}^{k} = (1+p-n)\,A_k r^p - (1+q-n)\,B_k r^q - \left[\frac{(3+\nu)(3-n)}{(2-p)(2-q)} - 1\right]\rho\omega^2 r^2 \tag{6}.$$

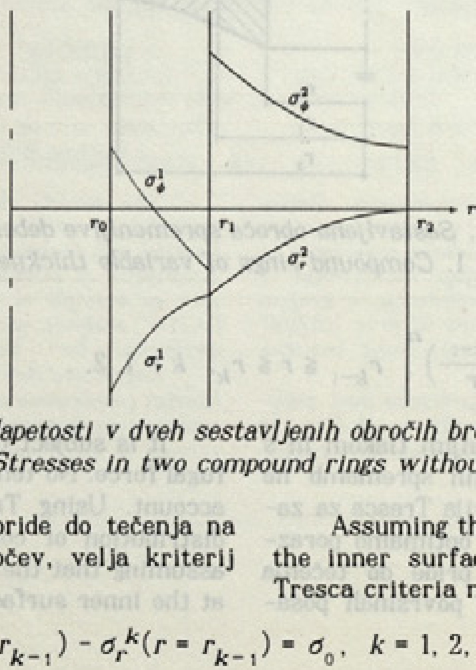
Ustrezni robni in vmesni pogoji so (sl. 2):

The corresponding boudary and continuity conditions are (fig.2):

$$\sigma_r^1(r=r_0) = -\,p_0 \tag{7}$$

$$\sigma_r^k(r=r_k) = \sigma_r^{k+1}(r=r_k),\quad k = 1, 2, \ldots, N-1 \tag{8},$$

$$\sigma_r^N(r=r_N) = 0 \tag{9}.$$



Sl. 2. *Napetosti v dveh sestavljenih obročih brez vrtenja.*
Fig. 2. *Stresses in two compound rings without rotation.*

Ker predpostavljamo, da pride do tečenja na notranjih površinah vseh obročev, velja kriterij Tresca:

Assuming that yield occurs simultaneously at the inner surfaces of each ring, the following Tresca criteria need to be applied as well:

$$\sigma_{\Phi}^{k}(r=r_{k-1}) - \sigma_r^k(r=r_{k-1}) = \sigma_0,\quad k = 1, 2, \ldots, N \tag{10}.$$

Iz teh $2N+1$ pogojev lahko izračunamo vrednosti p_0, A_k in B_k, tako da dobimo:

From the above set of $2N+1$ conditions values of p_0, A_k and B_k can be determined uniquely:

$$B_k = \frac{1}{n-q}\left\{\sigma_0 + \left[\frac{(3+\nu)(2-n)}{(2-p)(2-q)} - 1\right]\rho\omega^2 r_{k-1}^2 + (n-p)\,A_k r_{k-1}^p\right\} r_{k-1}^{-q} \qquad (11)$$

in

and

$$A_k = \sum_{j=0}^{N-k} \frac{(p-q)^{N-k-j}\, r_{N-j}^{-p}}{\prod_{m=0}^{N-k-j}\left[n-q-(n-p)\left(\frac{r_{N-1-j-m}}{r_{N-j-m}}\right)^{p-q}\right]}\left(\sigma_0\left[\left(\frac{r_{N-j}}{r_{N-1-j}}\right)^q - 1 + \delta_{0j}\right] + \right.$$

$$+\rho\omega^2\left\{\left[\frac{(3+\nu)(2-n)}{(2-p)(2-q)} - 1\right] r_{N-1-j}^2\left[\left(\frac{r_{N-j}}{r_{N-1-j}}\right)^q - \left(\frac{r_{N-j}}{r_{N-1-j}}\right)^2\right]\left(1-\delta_{0j}\right) + \right.$$

$$\left.\left. + \frac{(3+\nu)(n-q)}{(2-p)(2-q)}\, r_N^{2-p}\,\delta_{0j}\right\}\right) \qquad (12),$$

kjer je $\delta_{0j} = 0$, če $j \neq 0$ (tj. $\delta_{00} = 1$), medtem ko iz enačbe (7) izhaja:

where $\delta_{0j} = 0$, if $j \neq 0$ (i.e. $\delta_{00} = 1$), while from eq. (7):

$$p_0 = -A_1 r_0^p + B_1 r_0^q + \frac{3+\nu}{(2-p)(2-q)}\rho\omega^2 r_0^2 \qquad (13).$$

2. PRIMER DVEH OBROČEV

3. CASE OF TWO RINGS ONLY

Z $N = 2$ lahko prejšnje vrednosti zapišemo v obliki:

For $N = 2$ the above values are given by:

$$B_1 = \frac{1}{n-q}\left\{\sigma_0 + \left[\frac{(3+\nu)(2-n)}{(2-p)(2-q)} - 1\right]\rho\omega^2 r_0^2 + (n-p)\,A_1 r_0^p\right\} r_0^{-q} \qquad (14),$$

$$B_2 = \frac{1}{n-q}\left\{\sigma_0 + \left[\frac{(3+\nu)(2-n)}{(2-p)(2-q)} - 1\right]\rho\omega^2 r_1^2 + (n-p)\,A_2 r_1^p\right\} r_1^{-q} \qquad (15),$$

in

and

$$A_1 = \frac{\sigma_0\left[\left(\frac{r_1}{r_0}\right)^q - 1\right] + (p-q)A_2 + \rho\omega^2\left[\frac{(3+\nu)(2-n)}{(2-p)(2-q)} - 1\right]\left[\left(\frac{r_1}{r_0}\right)^q - \left(\frac{r_1}{r_0}\right)^2\right] r_0^2}{n-q-(n-p)(r_0/r_1)^{p-q}}\, r_1^{-p} \qquad (16),$$

$$A_2 = \frac{\sigma_0\left(\frac{r_2}{r_1}\right)^q + \rho\omega^2\left\{\left[\frac{(3+\nu)(2-n)}{(2-p)(2-q)} - 1\right]\left(\frac{r_1}{r_2}\right)^{2-q} + \frac{(3+\nu)(n-q)}{(2-p)(2-q)}\right\} r_2^2}{n-q-(n-p)(r_1/r_2)^{p-q}}\, r_2^{-p} \qquad (17)$$

oziroma / while

$$p_0 = \frac{\sigma_0}{n-q} - A_1 \frac{p-q}{n-q} + \rho\omega^2 r_0^2 \frac{1+p+\nu}{(2-p)(2-q)} \quad (18).$$

Kadar je $n = 0$, (tj. disk z nespremenljivo debelino), dobimo še preprostejše izraze:

For $n = 0$ (i.e. disk of constant thickness) the above simplifies to:

$$B_1 = \frac{\sigma_0}{2} r_0^2 - \frac{1-\nu}{8} \rho\omega^2 r_0^4 \quad (19),$$

$$B_2 = \frac{\sigma_0}{2} r_1^2 - \frac{1-\nu}{8} \rho\omega^2 r_1^4 \quad (20)$$

in / and

$$A_1 = \frac{\sigma_0}{2}\left[\left(\frac{r_0}{r_1}\right)^2 + \left(\frac{r_1}{r_2}\right)^2 - 1\right] + \frac{\rho\omega^2}{8} r_2^2 \left\{3 + \nu - (1-\nu)\left[\left(\frac{r_1}{r_2}\right)^4 + \left(\frac{r_0^2}{r_1 r_2}\right)^2 - \left(\frac{r_1}{r_2}\right)^2\right]\right\} \quad (21),$$

$$A_2 = \frac{\sigma_0}{2}\left(\frac{r_1}{r_2}\right)^2 + \frac{\rho\omega^2}{8} r_2^2 \left[3 + \nu - (1-\nu)\left(\frac{r_1}{r_2}\right)^4\right] \quad (22)$$

oziroma / while

$$p_0 = \frac{\sigma_0}{2}\left[2 - \left(\frac{r_1}{r_2}\right)^2 - \left(\frac{r_0}{r_1}\right)^2\right] -$$

$$- \frac{\rho\omega^2}{8} r_2^2 \left\{3 + \nu - (1-\nu)\left[\left(\frac{r_1}{r_2}\right)^4 + \left(\frac{r_0^2}{r_1 r_2}\right)^2 - \left(\frac{r_1}{r_2}\right)^2\right] + 2(1+\nu)\left(\frac{r_0}{r_2}\right)^2\right\} \quad (23).$$

3. OPTIMALNI RAZPORED

3. OPTIMUM DESIGN

Da dobimo največjo mogočo vrednost mejnega tlaka v elastičnem območju, izničimo odvod enačbe (23):

In order to find the maximum value of elastic limit pressure, let in eq. (23)

$$\frac{\partial p_0}{\partial r_1} = 0 \quad (24),$$

kar da kubično enačbo:

rendering a cubic equation:

$$\left(\frac{1-\nu}{2} \rho\omega^2 / r_2^2\right) r_1^6 - \left(\sigma_0 / r_2^2 + \frac{1-\nu}{4} \rho\omega^2\right) r_1^4 + \left(\sigma_0 - \frac{1-\nu}{4} \rho\omega^2 r_0^2\right) r_0^2 = 0 \quad (25),$$

ki jo razrešimo na iskano optimalno vrednost r_1 s standardnimi postopki. V splošnejšem primeru ($n \neq 0$) pa namesto enačbe (25) dobimo bolj zapleten izraz, ki ga lahko razrešimo na iskano optimalno vrednost r_1 le numerično.

Brez vrtenja ($\omega = 0$) in z nespremenljivo debelino ($n = 0$) preprosteje dobimo:

which may be solved for optimum value of r_1 by standard procedure.

In the general case ($n \neq 0$) the corresponding expression to eq. (25) obtains a cumbersome form, which may be solved numerically for optimum value of r_1.

In the case without rotation ($\omega = 0$) and constant thickness ($n = 0$), it simply follows that:

$$r_1^4 = r_0^2 r_2^2 \quad (26)$$

ali

or

$$r_1 = \sqrt{r_0 r_2} \quad (27).$$

Kakor izhaja iz vira [2], lahko to enačbo (27) zapišemo v obliki:

As shown in [2], eq. (28) may be written as:

$$r_{k-1}/r_k = r_k/r_{k+1} = \text{const} \quad (28).$$

4. POPOLNA PLASTIČNA PORUŠITEV

Če vrednost mejnega notranjega tlaka p_0 še bolj povečamo, se plastičnost razširi v notranjost vseh obročev. Ustrezna splošna elasto-plastična rešitev v vsakem obroču je:

4. ULTIMATE PLASTIC COLLAPSE

If value of internal pressure p_0 is further increased, the plastic zones develop from the inside in each of the rings. The general elasto-plastic solution for individual tubes is given by:

$$\sigma_r^k = C_k r^n + \frac{\sigma_0}{n} - \frac{\rho\omega^2 r^2}{2-n} \quad (29).$$

Z upoštevanjem pogojev (7), (8) in (9), dobimo velikost največjega mogočega notranjega tlaka, ki povzroči plastično porušitev:

Bearing in mind the existing boundary conditions (7), (8) and (9), the value of the ultimate plastic collapse pressure is:

$$p_u = \left(\frac{r_0}{r_N}\right)^n \left\{ \frac{\sigma_0}{n}\left[1 - \left(\frac{r_N}{r_0}\right)^n\right] - \frac{\rho\omega^2}{2-n} r_N^2 \left[1 - \left(\frac{r_0}{r_N}\right)^{2-n}\right]\right\} \quad (30).$$

Za $n = 0$ moramo rešitev (29) nadomestiti s:

However with $n = 0$, the solution (29) has to be replaced by:

$$\sigma_r^k = C_k + \sigma_0 ln(r) - \frac{1}{2}\rho\omega^2 r^2 \quad (31),$$

tako da je v tem primeru porušni tlak:

and the ultimate plastic collapse pressure is now:

$$p_u = \sigma_0 ln\left(\frac{r_N}{r_0}\right) - \frac{1}{2}\rho\omega^2 r_N^2\left[1 - \left(\frac{r_0}{r_N}\right)^2\right] \quad (32).$$

Dobljeni rezultat pomeni, da sestavljanje diska iz obročev sicer zadrži začetek plastifikacije, nima pa vpliva na končno vrednost porušnega tlaka, saj bi enak rezultat kakor (30) oziroma (32) dobili tudi z obravnavo enovitega diska z enakim razponom $r_N - r_0$.

These findings mean that the compounding of rings which postponed the yield initialisation, does not affect the ultimate collapse, since the same results (30) and (32) are also valid for a single ring of the equal total span $r_N - r_0$.

5. SKLEP

V prispevku smo razširili znane obrazce, ki veljajo za mirujoč disk nespremenljive debeline [2] oziroma za enovit disk spremenljive debeline [3] (hiperbolična oblika, ki jo uporabljajo v turbinah), z določitvijo največje vrednosti mejnega tlaka na notranji odprtini, ob upoštevanju kriterija Tresca. Če pa bi hoteli uporabiti Misesov kriterij, moramo enačbo (10) zamenjati z obrazcem:

5. CONCLUSION

Known expressions for compound disks of constant thickness [2] have been blended with results [3] of a single disk of variable thickness (hyperbolic shape used in turbine design). Optimum distribution of consecutive rings has been determined, allowing the maximum value of the limit pressure at the internal orifice of the innermost ring, assuming the Tresca yield criterion. If the Mises case is to be considered, (10) is to be replaced by

$$\sigma_\Phi^k \sigma_\Phi^k - \sigma_\Phi^k \sigma_r^k + \sigma_r^k \sigma_r^k = \sigma_0 \sigma_0 \qquad (33).$$

Temperaturno porazdelitev bi prav tako lahko upoštevali [4], [5].

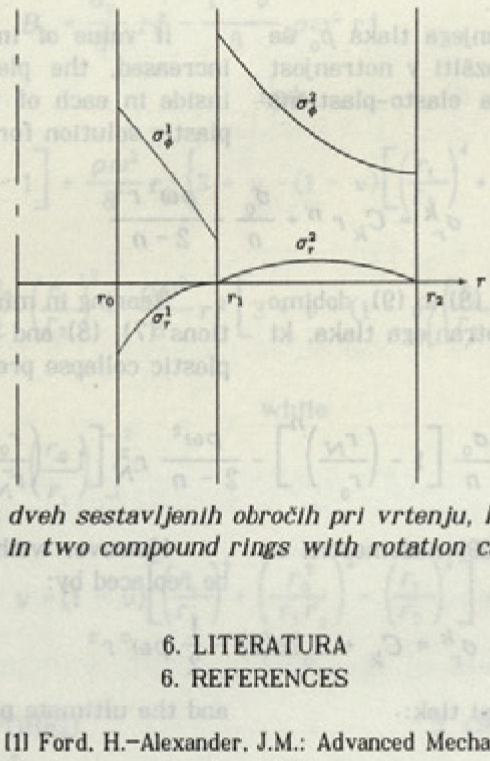
Velja tudi omeniti, da izpeljani obrazci ne veljajo, če pride do ločitve med obroči ob ustreznem povečanju vrtilne hitrosti.

V primeru na sliki 3 pride do ločitve pri 72,88 s^{-1} (n = 0, σ_0 = 300 MPa, ρ = 7850 kg/m^3, r_2/r_0 = 4). Če je $n > 0$, pride zaradi manjših centrifugalnih sil do ločitve pri večji kotni hitrosti.

Temperature gradient may also be considered [4], [5].

It has to be noted that developed formulae are not valid beyond the separation of rings, caused by certain increase of rotation speed.

For the case presented on fig. 3, the separation speed obtained was 72.88 s^{-1} (n = 0, σ_0 = 300 MPa, ρ =7850 kg/m^3, r_2/r_0 = 4). If $n > 0$, higher separation speed is observed due to lower centrifugal forces.



Sl. 3 *Napetosti v dveh sestavljenih obročih pri vrtenju, ki povzroči ločitev.*
Fig. 3. *Stresses in two compound rings with rotation causing separation.*

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