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Gretje tekočine zaradi viskoznega trenja

Heating of Fluids by Viscous Friction

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0. UVOD

V inženirski praksi imamo pogosto opravka s pretakanjem različnih tekočin po ceveh in kanalih. Pri toplotni analizi problema nas zanima predvsem temperaturno polje v tekočini. S poznavanjem tega lahko izračunamo toplotni fluks oziroma koeficient prestopa toplote, upoštevamo vpliv temperature na trdnost konstrukcije, preizkušamo vpliv različnih izolacijskih materialov.

Problem opišemo s sistemom parcialnih diferencialnih enačb, katerega izpeljemo z upoštevanjem ohranitvenih zakonov. Vendar inženirsko zanimivi primeri zaradi nelinearnosti enačb, robnih pogojev in geometrijske oblike niso analitično rešljivi, tako da moramo uporabiti eno izmed numeričnih metod.

V izpeljanih enačbah je navadno veliko členov, ki pa niso zmeraj vsi enako pomembni. Včasih kakšen člen zaradi majhnega vpliva zanemarimo in tako poenostavimo računski postopek. Takšen je npr. člen, ki v energijski enačbi pomeni viskozno trenje. V primeru močno viskozne tekočine ali pri velikih spremembah hitrosti, moramo upoštevati generacijo toplote zaradi notranjega trenja. Drugače lahko ta člen zanemarimo.

Navadno vsak tak člen vnese v numerično shemo določeno nelinearnost, kar se pozna pri stabilnosti sheme. Torej je treba vso skrb posvetiti modeliranju teh členov.

1. OSNOVNE ENAČBE

V ustaljenih razmerah je termo- in hidrodinamično stanje viskozne nestisljive tekočine opisano z ohranitvenimi zakoni:

0. INTRODUCTION

The case of various fluid flows in tubes and channels is encountered in engineering practice. Within thermal problem analysis, the temperature distribution in the fluid is of great importance. The thermal flux and heat transfer coefficient can be determined, bearing in mind the influence of temperatures on the strength of a structure and taking into account insulation materials.

The postulated problem is described by a system of partial differential equations which follow from the conservation balances. However, the major engineering problems, due to nonlinear equations, boundary conditions and geometrical shape, cannot be solved analytically and some numerical method has to be applied.

In the derived equations there are usually several terms, which may not be equally important. Some can therefore be neglected because of their small influence, thus simplifying the computational procedure. Such, e.g., is the term in the energy equation representing viscous friction. For highly viscous fluids and large changes of velocity, the heat generation due to internal friction has also to be considered. In other cases, this term is negligible.

Each term usually introduces a certain nonlinearity into the numerical scheme, which affects the stability of the scheme. Modelling of these terms has thus to be considered carefully.

1. GOVERNING EQUATIONS

In steady thermal and hydrodynamic conditions, the viscous incompressible fluid is described by the conservation laws:

— masa — mass

$$\nabla \boldsymbol{v} = 0 \tag{1}$$

— gibalna količina — momentum

$$(\boldsymbol{v}\nabla)\boldsymbol{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{v} + \boldsymbol{f} \tag{2}$$

— entalpija — enthalpy

$$(\boldsymbol{v}\nabla)T = a\,\nabla^2 T + \frac{\nu}{c_p}\,\Phi(\boldsymbol{v}) \tag{3}$$

V enačbah so spremenljivke: hitrost $\boldsymbol{v}$, tlak p, temperatura T, medtem ko pomeni $\boldsymbol{f}$ gostoto volumskih sil (navadno je to težnost) in $\Phi(\boldsymbol{v})$ Rayleighevo trosilno funkcijo. Snovske lastnosti: gostota (ρ), dinamična viskoznost (η), kinematična viskoznost ($\nu = \eta/\rho$), toplotna prevodnost (λ), specifična toplota (c_p) in difuzivnost ($a = \lambda/\rho c_p$) so konstantne.

Člen s $\Phi(\boldsymbol{v})$ v enačbi (3) pomeni vir toplote zaradi viskoznega trenja. Kadar notranje trenje ni izrazito, ga v enačbi ne upoštevamo. Rayleigheva trosilna funkcija je za ravninsko gibanje definirana z izrazom:

$$\Phi(\boldsymbol{v}) = 2\left(\left(\frac{\partial v_x}{\partial x}\right)^2 + \left(\frac{\partial v_y}{\partial y}\right)^2\right) + \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)^2 \tag{4}$$

Vidimo, da se v enačbi (4) pojavljajo odvodi hitrostnega polja. Poiskati je treba primeren postopek za njihov izračun. Ti odvodi namreč ne spadajo med osnovne spremenljivke, ki se pojavljajo v ohranitvenih enačbah za gibanje tekočine.

Pri numeričnem postopku metode robnih elementov enačbe (1), (2), (3) z Greenovimi identitetami prevedemo v sistem robno-območnih integralskih enačb, ki jih rešimo v diskretni obliki. Uporabimo vrtinčno-hitrostno formulacijo, kar pomeni, da uvedemo novo spremeljivko $\boldsymbol{w} = \nabla \times \boldsymbol{v}$, ter tako razdelimo reševanje na kinematski in kinetski del. Podrobneje je postopek opisan v [5], [4] ali [2]. Končni rezultat je matrični sistem $[A(x)]\{x\} = \{b(x)\}$, ki je nelinearen in ga rešujemo iterativno. Ko je hitrostno polje znano (za predpisane robne pogoje), rešimo še toplotni del sistema za dane robne pogoje.

2. IZRAČUN ODVODOV

Odvode hitrosti $\partial v_x/\partial x$; $\partial v_x/\partial y$; $\partial v_y/\partial x$; $\partial v_y/\partial y$, ki jih potrebujemo pri izračunu funkcije $\Phi(\boldsymbol{v})$, lahko dobimo iz znanega hitrostnega polja na dva načina. V prvem primeru postopamo kakor pri metodi končnih elementov. Gradient funkcije u,

Where the variables are: velocity $\boldsymbol{v}$, pressure p, temperature T, while $\boldsymbol{f}$ represents the density of volume forces (usually gravity) and $\Phi(\boldsymbol{v})$ the Rayleigh dissipation function. Material properties: density (ρ), dynamic viscosity (η), kinematic viscosity ($\nu = \eta/\rho$), heat conductivity (λ), specific heat (c_p) and diffusivity ($a = \lambda/\rho c_p$) are constants.

The term $\Phi(\boldsymbol{v})$ in equation (3) means heat source due to viscous friction. When internal friction is not important, it may be neglected in the equation. The Rayleigh dissipation function is defined for plane motion by the following expression:

It can be seen that derivatives of the velocity field are present in equation (4). An adequate method has to be found to evaluate them. These derivatives are not among the simple variables in the conservation equations of fluid motion.

Using the numerical procedure of the boundary element method, equations (1), (2), (3) are transformed by Green's identities into a system of boundary-domain integral equations, which may be solved in a discretised form using the vorticity-velocity formulation, meaning that by the introduction of a new variable $\boldsymbol{w} = \nabla \times \boldsymbol{v}$, the problem is divided into its kinematic and kinetic parts. The details are given in [5], [4], or [2]. The final result is a matrix system $[A(x)]\{x\} = \{b(x)\}$ which is nonlinear and has to be solved iteratively. Once the velocity distribution is determined (for prescribed boundary conditions), the thermal part of the system is also solved for given boundary conditions.

2. DETERMINATION OF DERIVATIVES

Velocity derivatives $\partial v_x/\partial x$; $\partial v_x/\partial y$; $\partial v_y/\partial x$; $\partial v_y/\partial y$, needed for the evaluation of function $\Phi(\boldsymbol{v})$ may be obtained from the known velocity field in two ways. In the first case it may ki jo zapišemo kot produkt interpolacijskih polinomov in vozliščnih vrednosti $u = \{\varphi\}^T\{U\}$, izračunamo z odvodi interpolacijskih polinomov:

be done as within the finite element method. The gradient of function u written as a product of interpolation polynomials and nodal values $u = \{\varphi\}^T\{U\}$, is computed by the use of derivatives of the interpolation polynomials:

$$\nabla u = \nabla\{\varphi\}^T\{U\} = \left[J^{-1}\right]\begin{bmatrix}\{\partial\varphi/\partial\xi\}^T \\ \{\partial\varphi/\partial\eta\}^T\end{bmatrix}\{U\} \tag{5}$$

Matrika $\left[J^{-1}\right]$ je inverzna Jacobijeva matrika transformacije iz globalnega koordinatnega sistema (x, y) v lokalni sistem (ξ, η), ki je vezan na celico. Ker je znano, da je numerično odvajanje slab postopek, se skušamo izračunu odvodov hitrosti po tej poti izogniti.

Druga pot vodi prek odvajanja integralskih enačb. Kinematski enačbi:

The matrix $\left[J^{-1}\right]$ represents the inverse Jacobian transformation matrix from a global coordinate system (x, y) into the local one (ξ, η) which is valid for a cell. Since it is well known that the numerical derivative evaluation is a poorly conditioned procedure, computation by the above technique should be avoided.

The second option is via the derivatives of integral equations. The kinematic equations:

$$c(\xi)\,v_x(\xi) + \int_\Gamma v_x(s)\frac{\partial u^*(\xi,s)}{\partial n(s)}\mathrm{d}\Gamma = \int_\Gamma v_y(s)\frac{\partial u^*(\xi,s)}{\partial t(s)}\mathrm{d}\Gamma - \int_\Omega w(s)\,\frac{\partial u^*(\xi,s)}{\partial y_s}\mathrm{d}\Omega \tag{6}$$

$$c(\xi)\,v_y(\xi) + \int_\Gamma v_y(s)\frac{\partial u^*(\xi,s)}{\partial n(s)}\mathrm{d}\Gamma = -\int_\Gamma v_x(s)\frac{\partial u^*(\xi,s)}{\partial t(s)}\mathrm{d}\Gamma + \int_\Omega w(s)\,\frac{\partial u^*(\xi,s)}{\partial x_s}\mathrm{d}\Omega \tag{7}$$

opisujeta zvezo med hitrostnim in vrtinčnim poljem. Če poiščemo gradient teh dveh enačb, dobimo odvode komponent hitrosti:

describe the connection between the velocity and vorticity fields. Searching for the gradient of these two equations, derivatives of the velocity components are obtained:

$$\begin{aligned} c(\xi)\frac{\partial v_x(\xi)}{\partial x} &= b_{xx}(\xi)\,v_x(\xi) - \int_\Gamma v_x(s)\frac{\partial}{\partial x_\xi}\left(\frac{\partial u^*(\xi,s)}{\partial n(s)}\right)\mathrm{d}\Gamma + \\ &+ b_{xy}(\xi)\,v_y(\xi) - \int_\Gamma v_y(s)\frac{\partial}{\partial x_\xi}\left(\frac{\partial u^*(\xi,s)}{\partial t(s)}\right)\mathrm{d}\Gamma + \\ &+ c_{xy}(\xi)\,w(\xi) - \int_\Omega w(s)\frac{\partial}{\partial x_\xi}\left(\frac{\partial u^*(\xi,s)}{\partial y_s}\right)\mathrm{d}\Omega \end{aligned} \tag{8}$$

$$\begin{aligned} c(\xi)\frac{\partial v_x(\xi)}{\partial y} &= b_{yx}(\xi)\,v_x(\xi) - \int_\Gamma v_x(s)\frac{\partial}{\partial y_\xi}\left(\frac{\partial u^*(\xi,s)}{\partial n(s)}\right)\mathrm{d}\Gamma + \\ &+ b_{yy}(\xi)\,v_y(\xi) - \int_\Gamma v_y(s)\frac{\partial}{\partial y_\xi}\left(\frac{\partial u^*(\xi,s)}{\partial t(s)}\right)\mathrm{d}\Gamma + \\ &+ c_{yy}(\xi)\,w(\xi) - \int_\Omega w(s)\frac{\partial}{\partial y_\xi}\left(\frac{\partial u^*(\xi,s)}{\partial y(s)}\right)\mathrm{d}\Omega \end{aligned} \tag{9}$$

$$
\begin{aligned}
c(\xi)\frac{\partial v_y(\xi)}{\partial x} = {} & b_{xy}(\xi)\, v_y(\xi) - \int_\Gamma v_y(s)\frac{\partial}{\partial x_\xi}\left(\frac{\partial u^*(\xi,s)}{\partial n(s)}\right) d\Gamma \, + \\
& + b_{xx}(\xi)\, v_x(\xi) - \int_\Gamma v_x(s)\frac{\partial}{\partial x_\xi}\left(\frac{\partial u^*(\xi,s)}{\partial t(s)}\right) d\Gamma \, + \\
& + c_{xx}(\xi)\, w(\xi) + \int_\Omega w(s)\frac{\partial}{\partial x_\xi}\left(\frac{\partial u^*(\xi,s)}{\partial x_s}\right) d\Omega
\end{aligned}
\qquad (10)
$$

$$
\begin{aligned}
c(\xi)\frac{\partial v_x(\xi)}{\partial y} = {} & b_{yy}(\xi)\, v_y(\xi) - \int_\Gamma v_y(s)\frac{\partial}{\partial y_\xi}\left(\frac{\partial u^*(\xi,s)}{\partial n(s)}\right) d\Gamma \, + \\
& + b_{yx}(\xi)\, v_x(\xi) - \int_\Gamma v_x(s)\frac{\partial}{\partial y_\xi}\left(\frac{\partial u^*(\xi,s)}{\partial t(s)}\right) d\Gamma \, + \\
& + c_{yx}(\xi)\, w(\xi) - \int_\Omega w(s)\frac{\partial}{\partial y_\xi}\left(\frac{\partial u^*(\xi,s)}{\partial x_s}\right) d\Omega
\end{aligned}
\qquad (11)
$$

Ker so integrandi singularni, ko se izvorna točka ξ ujema z integracijsko točko s, točko ξ ogradimo iz območja s krogom polmera ε. Ko izvedemo limitni postopek $\varepsilon \to 0$ se pojavijo prosti členi c, $b_{I,J}$, $c_{I,J}$, ki so odvisni le od geometrijske oblike. Vendar jih dejansko ni treba poznati, saj jih skupaj z diagonalnimi elementi matrik izračunamo z uporabo partikularnih rešitev. Podrobneje je to opisano v [3].

Z diskretizacijo območja zapišemo sistem (8) do (11) v matrični obliki za ekspliciten izračun odvodov komponent hitrosti:

Since the integrands are singular when the source point ξ coincides with integration point s, point ξ has to be fenced from the domain by a circle of radius r. Performing the limit process $\varepsilon \to 0$ the free terms c, $b_{I,J}$, $c_{I,J}$ appear, which are dependent on the geometry only. They do not need to be known, since they may be evaluated from diagonal terms of matrices by the use of particular solutions. This is described in detail in [3].

Discretising the domain, system (8) to (11) is written in matrix form for explicit calculation of velocity component derivatives:

$$\left\{c(\xi)\frac{\partial v_x(\xi)}{dx}\right\} = -\left[H_{xn}\right]\left\{V_x\right\} + \left[H_{xt}\right]\left\{V_y\right\} - \left[D_{xy}\right]\left\{W\right\} \qquad (12)$$

$$\left\{c(\xi)\frac{\partial v_x(\xi)}{dy}\right\} = -\left[H_{yn}\right]\left\{V_x\right\} + \left[H_{yt}\right]\left\{V_y\right\} - \left[D_{yy}\right]\left\{W\right\} \qquad (13)$$

$$\left\{c(\xi)\frac{\partial v_y(\xi)}{dx}\right\} = -\left[H_{xn}\right]\left\{V_y\right\} + \left[H_{xt}\right]\left\{V_x\right\} - \left[D_{xx}\right]\left\{W\right\} \qquad (14)$$

$$\left\{c(\xi)\frac{\partial v_y(\xi)}{dy}\right\} = -\left[H_{yn}\right]\left\{V_y\right\} + \left[H_{yt}\right]\left\{V_x\right\} - \left[D_{yx}\right]\left\{W\right\} \qquad (15)$$

Matrike [H] in [D] so sestavljene iz vozliščnih prispevkov ustreznih integralov po elementih oziroma celicah.

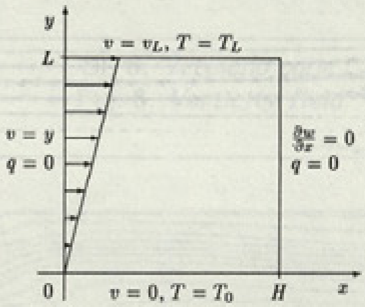
Matrices $[H]$ and $[D]$ are composed of nodal contributions of corresponding integrals by elements and cells respectively.

3. TESTNI PRIMER

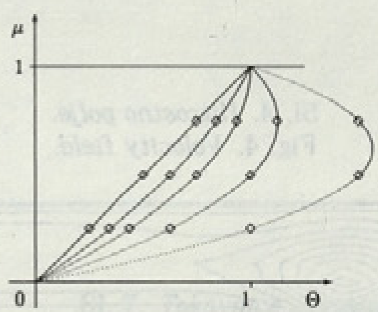
Preverimo učinkovitost predstavljene sheme na primeru, ki je analitično rešljiv. Tak zgled je Couettov tok, glej [1], kjer je porazdelitev hitrosti po višini linearna. Za robne pogoje in geometrijske podatke, prikazane na sliki 1, je temperaturni profil dan z enačbo za normirane vrednosti $\Theta = \mu(1 + \frac{1}{2} Pr\, E(1 - \mu))$, pri čemer so brezdimenzijska števila $\Theta = (T - T_0)/(T_L - T_0)$, $\mu = y/L$, $Pr = \nu/a$ in $E = v_L^2/(c_p(T_L - T_0))$. Odstopanje numeričnih rezultatov od analitičnih za izbrani primer je v okviru natančnosti računanja. Potek temperature za različna števila PrE prikazuje slika 2.

3. TEST CASE

The proposed scheme can be tested with a case which has an analytical solution. Such is the Couette flow, ref. [1], with linear distribution of velocity by height. Boundary conditions and geometrical data, shown in Fig. 1, yield a temperature profile in the form of normalized values $\Theta = \mu(1 + \frac{1}{2} Pr\, E(1 - \mu))$, where nondimensional values are $\Theta = (T - T_0)/(T_L - T_0)$, $\mu = y/L$, $Pr = \nu/a$ and $E = v_L^2/(c_p(T_L - T_0))$. Deviations between numerical and analytical results for the selected case are within computational errors. The temperature distribution for the variable value of $Pr\,E$ is given in fig. 2.



Sl. 1. *Geometrijski podatki in robni pogoji za Couettov tok.*
Fig. 1. *Geometrical data and boundary conditions of Couette flow.*



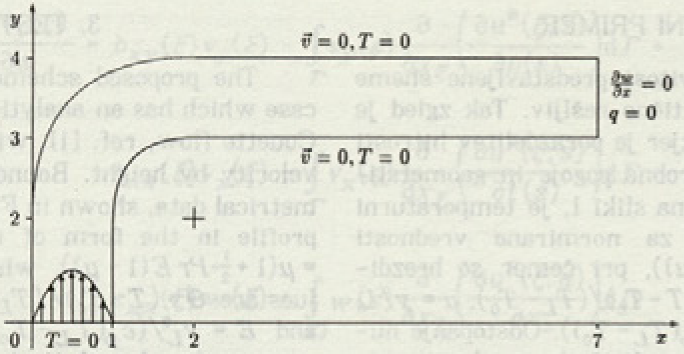
Sl. 2. *Temperaturni profili pri PrE = 0, 1, 2, 4, 8 za Couettov tok.*
Fig. 2. *Temperature profiles for PrE = 0, 1, 2, 4, 8 of Couette flow.*

4. TOK MOČNO VISKOZNE TEKOČINE V UKRIVLJENEM KANALU

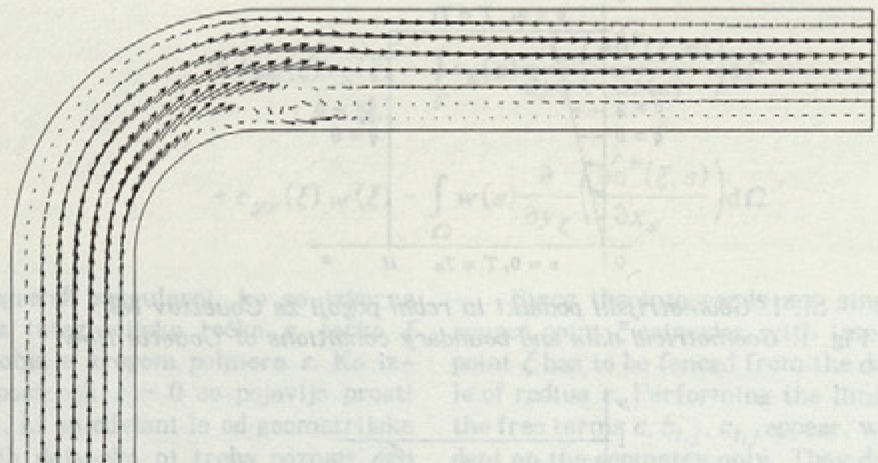
Kot analitično nerešljiv primer si oglejmo gretje tekočine zaradi notranjega trenja. V ukrivljen kanal vstopa tekočina s temperaturo $T = 0$ in razvitim laminarnim hitrostnim profilom (parabola). Geometrijski podatki in robni pogoji so prikazani na sliki 3. Zaradi zasuka kanala za 90° in velike hitrosti tekočine (Re = 500), se po prehodu v vodoravno smer pojavi območje povratnih tokov ob spodnji steni. Hitrostno polje je prikazano na sliki 4. Zaradi velikih sprememb hitrosti na tem območju (veliki odvodi $\boldsymbol{v}$) je velik tudi vpliv viskoznega trenja in s tem povezano gretje tekočine zaradi notranjega trenja. Slika 5 prikazuje

4. FLOW OF HIGHLY VISCOUS FLUID IN A CURVED CHANNEL

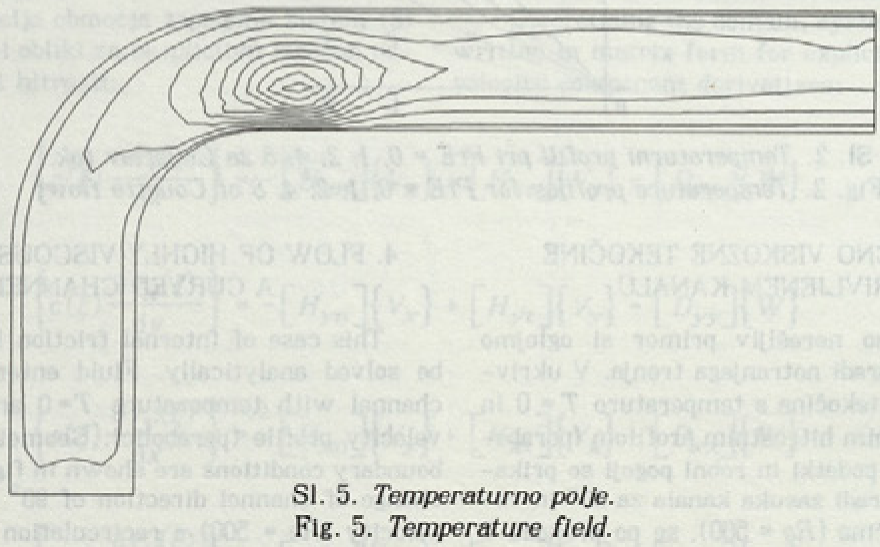
This case of internal friction heating cannot be solved analytically. Fluid enters the curved channel with temperature $T = 0$ and a developed velocity profile (parabolic). Geometrical data and boundary conditions are shown in fig. 3. Due to the change of channel direction of 90° and high fluid velocity (Re = 500) a recirculation zone develops at the horizontal part of the channel close to the lower wall. The velocity field is shown in Fig. 4. Due to large changes in velocity in this part (large derivatives of $\boldsymbol{v}$) viscous dissipation has a major influence and causes heating of the fluid through internal friction. Fig. 5 shows the temperature



Sl. 3. *Geometrijski podatki in robni pogoji za tok tekočine v ukrivljenem kanalu (Re = 500, ρν = 8).*
Fig. 3. *Geometrical data and boundary conditions of fluid flow in curved channel (Re = 500, ρν = 8).*



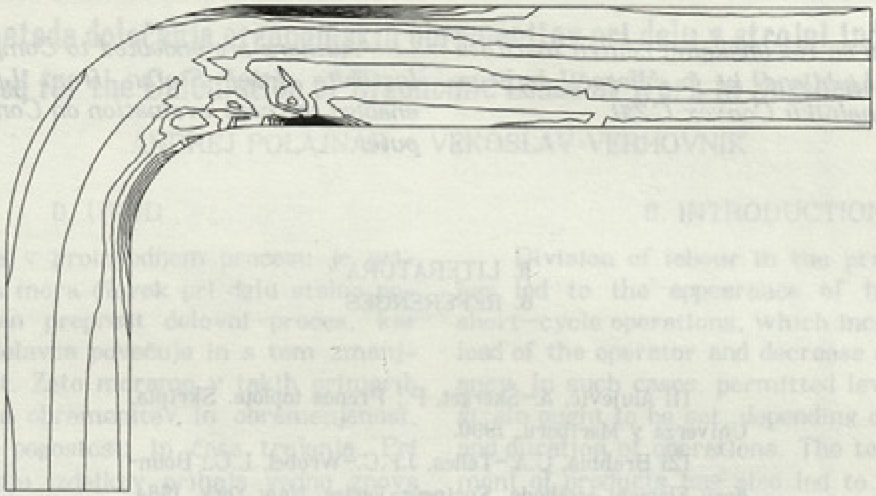
Sl. 4. *Hitrostno polje.*
Fig. 4. *Velocity field.*



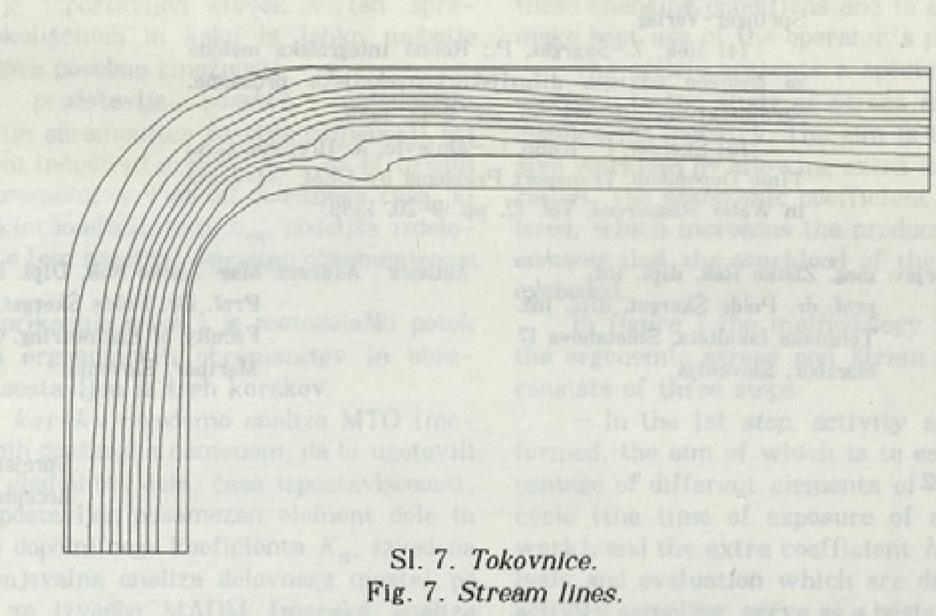
Sl. 5. *Temperaturno polje.*
Fig. 5. *Temperature field.*

temperaturno polje v tekočini. Mesto najvišje temperature se ujema z največjimi spremembami hitrosti. Slika 6 prikazuje vrtinčno polje, na sliki 7 pa so prikazane tokovnice za obravnavani primer.

field in the fluid. The location of the highest temperature coincides with the largest velocity changes. Fig. 6 shows the vorticity field, while the stream lines of the considered case are given in fig. 7.



Sl. 6. *Vrtinčno polje.*
Fig. 6. *Vorticity field.*



Sl. 7. *Tokovnice.*
Fig. 7. *Stream lines.*

5. SKLEP

V članku je predstavljena numerična shema reševanja toplotnih in hidrodinamičnih problemov ravninskega gibanja nestisljive tekočine, pri kateri ne smemo zanemariti vpliva viskoznega trenja. Posebna pozornost je namenjena modeliranju tega člena, saj v numerično shemo vnaša močno nelinearnost. Preizkus z analitično rešljivim primerom potrjuje učinkovitost uporabljenega numeričnega postopka.

5. CONCLUSION

A numerical scheme of thermal and hydrodynamic problems of plane moving of incompressible fluid has been presented for cases when viscous dissipation cannot be neglected. Special attention is paid to the modelling of this term, since it bears a strong nonlinearity to the numerical scheme. The test case with an existing analytical solution confirms the effectiveness of the proposed numerical procedure.

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6. LITERATURA
6. REFERENCES

[1] Alujevič, A.–Škerget, P.: Prenos toplote. Skripta, Univerza v Mariboru, 1990.

[2] Brebbia, C.A.–Telles, J.F.C.–Wrobel, L.C.: Boundary Element Methods. Springer-Verlag, New York, 1984.

[3] Guiggiani, M.–Krishnasamy, G.–Rizzo, F.J.–Rudolphi, T.J.: Boundary Integral Methods. Proceedings of the IABEM Symposium, Rome, October 15–19, 1990. Springer-Verlag.

[4] Rek, Z.–Škerget, P.: Robna integralska metoda za časovno odvisne difuzijsko-konvekcijske probleme. Strojniški vestnik, vol. 35, str. 9–12, 1989.

[5] Škerget, P.–Kuhn, G.–Alujevič, A.–Brebbia, C.A.: Time Dependent Transport Problems by BEM. Advances in Water Resources, Vol. 12, pp. 9–20, 1989.

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