Structural Optimization of an Engine Crankshaft-Bearing System Based on Deformation Coordination Analysis

S1. Detailed Derivation of Deformation Compatibility Equation

S1.1 Beam Idealization

The crankshaft is modeled as a continuous Euler–Bernoulli beam resting on seven elastic supports. Each span between supports is denoted as L_i.

At support i, the rotation angle results from superposition of:

- Bending moment at left support $M_{i\text{-}1}$
- Bending moment at right support M_i
- Concentrated load P_i
- Uniform distributed load q
- Equivalent bearing displacement ΔR_i

S1.2 Rotation Induced by Individual Loads

(1) Rotation caused by left bending moment

$$\theta_i^{(M_{i\text{-}1})}=\frac{M_{i\text{-}1}L_i}{6EI} \tag{S1}$$

(2) Rotation caused by right bending moment

$$\theta_i^{(M_i)}=-\frac{M_iL_i}{3EI} \tag{S2}$$

(3) Rotation caused by concentrated force

$$\theta_i^{(P)}=\frac{P_iL_i^2}{6EI} \tag{S3}$$

(4) Rotation caused by uniform distributed load

$$\theta_i^{(q)}=\frac{qL_i^3}{24EI} \tag{S4}$$

(5) Rotation caused by equivalent bearing displacement

$$\theta_i^{(\Delta R)} = \frac{\Delta R_i}{L_i} \quad \text{(S5)}$$

S1.3 Superposition and Continuity Condition

Total rotation at support i:

$$\theta_i = \sum_k \theta_i^{(k)} \quad \text{(S6)}$$

Continuity condition of the continuous beam:

$$\theta_i^{(left)} = \theta_i^{(right)} \quad \text{(S7)}$$

After arranging terms:

$$K_{i-1}\theta_{i-1} + K_i\theta_i + K_{i+1}\theta_{i+1} = F_i \quad \text{(S8)}$$

S2. Matrix Formulation of Support Reaction

Combining equilibrium equations for each span:

$$R = [A]\theta \quad \text{(S9)}$$

Where:

$$[A] = [a_{11} \quad a_{12} \quad \cdots \quad a_{1n} \; a_{21} \quad a_{22} \quad \cdots \quad a_{2n} \vdots \quad \vdots \quad \ddots \quad \vdots \; a_{n1} \quad a_{n2} \quad \cdots \quad a_{nn}]$$

Matrix coefficients depend on:

- Span length L_i
- Elastic modulus E
- Section moment of inertia I

S3. Geometric Series Expansion for Oil-Film Approximation

From the oil-film displacement relation:

$$\Delta y_i = R_i \cdot f(\chi) \quad \text{(S10)}$$

Using geometric series expansion:

$$(1-t)^{-1} = 1 + t + t^2 + \ldots \quad (|t| < 1) \quad \text{(S11)}$$

Let:

$$\chi=\chi_0+\delta\chi \tag{S12}$$

Using Taylor expansion:

$$\Phi(\chi)\approx\Phi(\chi_0)+\Phi'(\chi_0)\delta\chi \tag{S13}$$

Higher-order infinitesimal terms are neglected.

S4. Mesh Convergence Study

Three mesh densities were tested:

Mesh Type	Elements	Nodes	Max Bearing Reaction (N)	Deviation (%)
Coarse	16208	28560	120950	5.8%
Medium	21379	36012	118420	Reference
Fine	25733	42980	114400	3.4%

Maximum deviation between medium and fine mesh: 3.4%

Thus, the medium mesh is selected.

S5. Detailed Hydrodynamic Lubrication Formulation

S5.1 Reynolds Equation Derivation

Starting from Navier–Stokes equations under thin-film assumptions:

- Laminar flow
- Isoviscous lubricant
- Negligible inertia
- Thin film

The Reynolds equation becomes:

$$\frac{\partial}{\partial x}\left(\mathrm{h}^3\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial z}\left(\mathrm{h}^3\frac{\partial p}{\partial z}\right)=6\mu U\frac{\partial \mathrm{h}}{\partial x} \tag{S14}$$

S5.2 Film Geometry

Film thickness expression:

$$\mathrm{h}(\theta)=c(1+\chi\cos\theta) \tag{S15}$$

Where:

- c = radial clearance
- χ = eccentricity ratio

S5.3 Bearing Load Integration

Bearing load:

$$W = \int_0^{\pi} p\,(\theta)\cos\theta, d\theta \qquad \text{(S16)}$$

Using Gumbel boundary condition:

$$p = 0 \quad \text{in cavitation region}$$

S5.4 Exponential Fitting Function

The bearing capacity factor is fitted as:

$$\Phi(\chi) = ae^{b\chi} + ce^{d\chi} \qquad \text{(S17)}$$

Fitting statistics:

- $R^2 = 0.998$
- RMSE = 0.012

S6. Model Limitations

1. Torsional deformation neglected.
2. Crankcase elasticity simplified.
3. Thermal effects not included.
4. Full transient TEHD not solved.

These simplifications are acceptable for quasi-static deformation-coordination analysis.