UDK 539.3/.4:519.61/.64

Napetosti v vzdolžno prerezanem rotirajočem votlem valju

Stresses in Hollow Rotating Cylinder with Longitudinal Split

MILAN BATISTA – FRANC KOSEL

Raziskava vpliva radialnega vzdolžnega prereza na napetostno stanje vrtečega se votlega valja je eden od temeljnih korakov pri raziskavi porušitve različnih rotirajočih mehanskih delov, npr. brusni koluti. Kritični pregled trenutnega stanja na področju teh raziskav je podan v članku [1]. V omenjenem članku so avtorji C. Bandara, M. Nikolich, in A. Strozzi podali primerjalni diagram izračuna največje napetosti, ki se pojavi na notranji strani vzdolžno prerezanega votlega valja, po različnih metodah, med katerimi so teorija ukrivljenih nosilcev, teorija ravnih nosilcev, ter po metodi končnih elementov (MKE). Ugotovili so, da se rešitvi, dobljeni z MKE in teorijo ukrivljenih nosilcev praktično ujemata, druge metode pa dajejo nekoliko nižje vrednosti. Do velikega razhajanja pride na področju majhnih razmerij polmerov. Medtem ko se po teoriji ukrivljenih nosilcev in MKE napetost od razmerja polmerov $\lambda = a/b \approx 0{,}15$ navzdol zopet zvečuje, pa se po preostalih dveh metodah zmanjšuje.

Namen tega prispevka je podati matematični opis napetostnega stanja z uporabo napetostne Airyjeve funkcije ter posebej poudariti izpolnjevanje robnega pogoja na prerezni ploskvi.

Investigation of the stress state in a rotating ring is one of the basic steps in studying the bursting mechanism in different rotating machine components and tolls, for example a grinding wheel. A critical survey of the state of the art in this field of research was given by Bandera C., M. Nikolich and A. Strozzi, 1993. In their paper, they compare the maximum circumferential stress which appears on the inner side of the longitudinal cross section planes of the hollow cylinder, the stresses being determined using several different methods: curved beam theory, straight beam theory and finite element method (FEM). They found that the maximum circumferential stresses determined by the straight beam theory and FEM are almost the same, whereas the other methods give lower values of maximum circumferential stresses. Large differences among the maximum circumferential stresses occur near the inner radii. By the curved beam theory and FEM, the maximum circumferential stresses increase for radii ration from 0.15 downwards but, using the other two methods the same stresses decrease at these radii ratia.

Our contribution presents a mathematical approach to determine the function of the stress state in the longitudinal section planes of the hollow cylinder, using the Airy function. Special emphasis is laid on the need to meet the boundary conditions on the longitudinal cross section planes of the hollow cylinder.

1 PREDPOSTAVKE IN OZNAČBE

Votel valj z zunanjim polmerom b in notranjim polmerom a, se vrti z nespremenljivo kotno hitrostjo ω. Vzdolž negativnega dela osi x, je vzdolžno prerezan (sl. 1). Predpostavimo, da je valj izdelan iz homogenega in izotropnega gradiva in da napetostno stanje ne preseže meje elastičnosti.

Zaradi boljše preglednosti in kasnejše primerjave rezultatov uvedimo naslednje brezdimenzijske polmere:

1 ASSUMPTIONS AND NOTATION

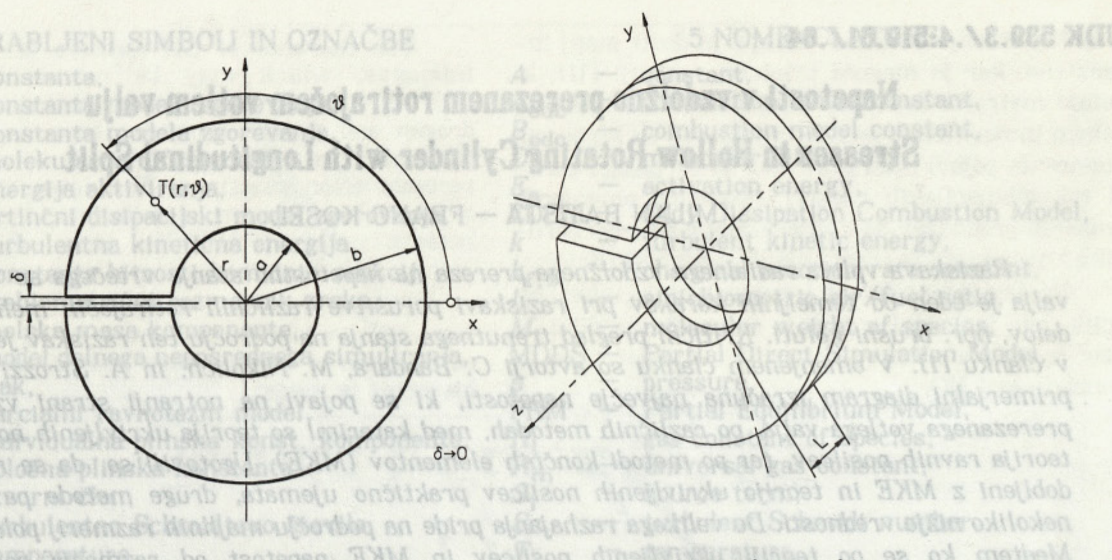
A hollow cylinder with an outer radius b, inner radius a and length L rotates at a constant angular speed. Along the negative side of the x-axis, the hollow cylinder is cut in a longitudinal direction, fig. 1. A further assumption is that it is made of homogeneous and isotropic material. The stress state does not exceed the elastic limit of the material.

For reasons of clarity of presentation and the possibility of final comparison of the results, we introduce the following dimensionless radii:

$$x = \frac{r}{b}\ , \quad \lambda = \frac{a}{b} \tag{1}$$

in napetosti: / and stresses:

$$\bar{\sigma}_r = \frac{\sigma_r}{\sigma_*}\ , \quad \bar{\sigma}_\vartheta = \frac{\sigma_\vartheta}{\sigma_*}\ , \quad \bar{\tau}_{r\vartheta} = \frac{\tau_{r\vartheta}}{\sigma_*}$$



Sl. 1. *Geometrijska oblika vzdolžno prerezanega votlega valja*
Fig. 1. *Geometry of a hollow cylinder with longitudinal split.*

pri čemer je normirana napetost σ^*:

where the normalized stress σ^* is equal to:

$$\sigma^* = \rho\omega^2 b^2 \qquad (3).$$

Z $\overline{F}$ označimo brezdimenzijsko napetostno Airyjevo funkcijo, za katero naj velja:

Let by $\overline{F}$ denote a dimensionless Airy stress function which reads:

$$F = \rho\omega^2 b^4\, \overline{F} \qquad (4).$$

2 DOLOČITEV AIRYJEVE NAPETOSTNE FUNKCIJE

2 DETERMINATION OF AIRY STRESS FUNCTION

Pri predpostavki, da je valj v navidezno-statičnem ravninskem napetostnem stanju, so komponente napetostnega tenzorja dane z naslednjimi enačbami [2]:

Let us assume that the hollow cylinder is in a quasi-static plane stress state. In this case, we can write the components of the stress tensor in the following form, [2]:

$$\overline{\sigma}_r = -\frac{3+\nu^*}{8}x^2 + \frac{1}{x^2}\frac{\partial^2\overline{F}}{\partial\vartheta^2} + \frac{1}{x}\frac{\partial\overline{F}}{\partial x} \qquad (5),$$

$$\overline{\sigma}_\vartheta = -\frac{1+3\nu^*}{8}x^2 + \frac{\partial^2\overline{F}}{\partial x^2} \qquad (6),$$

$$\overline{\tau}_{r\vartheta} = -\frac{\partial^2}{\partial x\,\partial\vartheta}\left(\frac{\overline{F}}{x}\right) \qquad (7),$$

kjer je:

where is:

$$\nu^* = \begin{cases} = \nu & \text{— ravninsko napetostno stanje (plane stress)} \\ = \dfrac{\nu}{1-\nu} & \text{— ravninsko deformacijsko stanje (plane strain)} \end{cases}$$

Prva člena v izrazih (5) in (6) sta partikularni rešitvi ravnotežnih in kompatibilnostne enačbe elastostatike.

Ker je vztrajnostna sila, ki se pojavi zaradi rotacije, edina obremenitev, na mejnih ploskvah valja ni napetosti. Na zunanji in notranji strani valja je torej:

The first two terms is expressions (5) and (6) are particular solutions of the equilibrium and compatibility equations of elasto - statics.

The inertia force caused by rotation is the only load, because the boundary surfaces of the hollow cylinder are not subjected to any outer stress vectors. From this follows that the stress tensors on the outer and inner side of the hollow cylinder vanish:

$$\bar{\sigma}_r = \bar{\tau}_{r\vartheta} = 0 \quad (x = \lambda \ {}^{\text{in}}_{\text{and}}\ x = 1) \tag{8}$$

na ploskvah, kjer je valj prerezan pa:

and on the surfaces of the longitudinal split:

$$\bar{\tau}_{r\vartheta} = 0 \quad (\vartheta = \pi) \tag{9}$$

$$\bar{\sigma}_{\vartheta} = 0 \quad (\vartheta = \pi) \tag{10}$$

Zaradi geometrijske in obremenitvene simetrije problema, saj ima vzdolžno prerezan valj samo eno simetrijsko os, je napetostna funkcija:

The treated problem of the longitudinal split planes of the hollow cylinder presents a problem of one symmetry, and the Airy stress function is:

$$\bar{F}(x,\vartheta) = \bar{F}_0(x) + \sum_{n=1}^{\omega} \bar{F}_n(x) \cos n\vartheta \tag{11}$$

Funkcije, ki se pojavljajo v zgornji vrsti so znane:

The functions which appear it the above sum are known:

$$\bar{F}_0(x) = C_1 \ln x + C_2 x^2 + C_3 x^2 \ln x \tag{12}$$

$$\bar{F}_1(x) = \frac{C_4}{x} + C_5 x^3 + C_6 x \ln x \tag{13}$$

$$\bar{F}_n(x) = A_n x^{-n} + B_n x^n + C_n x^{2-n} + D_n x^{n+2} = 0 \quad (n \geq 2) \tag{14}$$

pri čemer so C_i (i = 1 do 6) ter A_n, B_n, C_n in D_n integracijske konstante.

S funkcijo (11) iz enačb za izračun napetosti (5) do (7) po krajšem računanju dobimo:

where C_i (i = 1 to 6) while A_n, B_n, C_n and D_n are constants of integration.

After introducing function (11) into the expressions for stresses (5) to (7) and after some calculating, it follows:

$$\bar{\sigma}_r = -\frac{3+\nu^*}{8} x^2 + \frac{1}{x}\frac{d\bar{F}_0}{dx} + \sum_{1}^{\infty} \left[\frac{d}{dx}\left(\frac{\bar{F}_n}{x}\right) - (n^2-1)\frac{\bar{F}_n}{x^2}\right] \cos n\vartheta \tag{15}$$

$$\bar{\sigma}_{\vartheta} = -\frac{1+3\nu^*}{8} x^2 + \frac{d^2\bar{F}_0}{dx^2} + \sum_{1}^{\infty} \frac{d^2\bar{F}_n}{dx^2} \cos n\vartheta \tag{16}$$

$$\bar{\tau}_{r\vartheta} = \sum_{1}^{\infty} n \frac{d}{dx}\left(\frac{\bar{F}_n}{x}\right) \sin n\vartheta \tag{17}$$

Ker je v (17) $\sin(n\pi) = 0$ za vsako celo število n, je robni pogoj (9) avtomatično izpolnjen. Za izpolnitev robnih pogojev (8) pa mora biti v enačbi (15):

Because the expression $\sin(n\pi)$ in equation (17) is zero for any natural number n, the boundary condition (9) is automatically fulfilled. For the fulfillment of the boundary conditions (8) we have to use equation (15):

$$\frac{1}{x}\frac{d\bar{F}_0}{dx} = \frac{3+\nu^*}{8} x^2 \quad {}^{\text{za}}_{\text{for}}\ x = \lambda \ {}^{\text{in}}_{\text{and}}\ x = 1 \tag{18}$$

$$\frac{d}{dx}\left(\frac{\bar{F}_n}{x}\right) - (n^2-1)\frac{\bar{F}_n}{x^2} = 0 \quad (n \geq 1) \quad {}^{\text{za}}_{\text{for}}\ x = \lambda \ {}^{\text{in}}_{\text{and}}\ x = 1 \tag{19}$$

in v enačbi (17),

and equation (17):

$$\frac{d}{dx}\left(\frac{\bar{F}_n}{x}\right) = \frac{1}{x}\frac{d\bar{F}_n}{dx} - \frac{\bar{F}_n}{x^2} = 0 \quad (n \geq 1) \quad {}^{\text{za}}_{\text{for}}\ x = \lambda \ {}^{\text{in}}_{\text{and}}\ x = 1 \tag{20}$$

Iz pogojev (19) in (20) izhaja, da mora biti $(n^2-1)\bar{F}_n = 0$, torej mora veljati:

From conditions (19) and (20) we get $(n^2-1)\bar{F}_n = 0$, thus:

$$\frac{d}{dx}\left(\frac{\overline{F}_1}{x}\right) = 0 \quad \text{za/for } x = \lambda \text{ in/and } x = 1 \qquad (21),$$

$$\overline{F}_n(x) = \frac{d\overline{F}_n}{dx} = 0 \quad (n \geq 2) \quad \text{za/for } x = \lambda \text{ in/and } x = 1 \qquad (22).$$

Pogoj (22) je zaradi oblike funkcije (14) homogen sistem štirih enačb s štirimi neznankami. Ker je determinanta tega sistema različna od nič morajo biti:

Because of the shape of the functions (14) the condition (22) represents a homogeneous system of four equations with four unknowns. Since the determinant of coefficients of this system is different from zero:

$$A_n = B_n = C_n = D_n = 0 \quad (n \geq 2) \qquad (23),$$

kar pomeni, da velja:

which means that:

$$\overline{F}_n(x) \equiv 0 \quad (n \geq 2) \qquad (24).$$

Za rešitev problema ostaneta na voljo le še funkciji $\overline{F}_0$ in $\overline{F}_1$ s skupaj šestimi integracijskimi konstantami. Za izračun vrednosti štirih integracijskih konstant imamo robna pogoja (18) in (21). Z vstavljanjem funkcije (12) v pogoj (18) in funkcije (13) v pogoj (21) dobimo naslednji sistem štirih enačb:

For the solution of the problem we now have available only two functions $\overline{F}_0$ and $\overline{F}_1$ with a total of six integration constants. To determine the values of four integration constants we can use conditions (18) and (21). Introducing now function (12) into conditions (18) and function (13) into condition (21) we get the following system of four equations:

$$\frac{C_1}{\lambda^2} + 2C_2 + [2\ln(\lambda) + 1]C_3 = \frac{3+\nu^*}{8}\lambda^2$$

$$C_1 + 2C_2 + C_3 = \frac{3+\nu^*}{8} \qquad (25),$$

$$-2C_4 + 2\lambda^4 C_5 + \lambda^2 C_6 = 0$$

$$-2C_4 + 2C_5 + C_6 = 0$$

katerega rešitev je:

from which we get the values:

$$C_1 = -\frac{3+\nu^*}{8}\lambda^2 - \lambda^2 \ln\lambda\, A$$

$$2C_2 = \frac{3+\nu^*}{8}(1+\lambda^2) - [(1-\lambda^2)/2 - \lambda^2\ln\lambda]\, A \qquad (26),$$

$$2C_3 = (1-\lambda^2)\, A$$

in

and

$$2C_1 = \lambda^2 B, \quad 2C_2 = B, \quad C_6 = (1+\lambda^2)\, B \qquad (27),$$

pri čemer sta A in B novi integracijski konstanti.

Dobljena rešitev problema torej ustreza enačbam elastomehanike ter zadošča petim od skupaj šestih robnih pogojev. Z dano obliko rešitve se ne da natančno izpolniti robnega pogoja (10), zato sledi, da je, ne glede na vrednost integracijskih konstant A in B, na prereznih ploskvah napetost $\overline{\sigma}_\vartheta(x, \pi)$ različna od nič:

where A, B denote the new constants of integration. In this way we obtained a solution of the problem which is in agreement with the equations of elastomechanics and meets five out of a total of six boundary conditions. However, this kind of solution cannot meet the condition (10). Therefore, irrespective of the values of the constants of integration A and B, the circumferential stress tensor component on the longitudinal cut planes of the hollow cylinder is:

$$\overline{\sigma}_\vartheta(x, \pi) = -\frac{1+3\nu^*}{8}x^2 + \frac{d^2\overline{F}_0}{dx^2} - \frac{d^2\overline{F}_1}{dx^2} \neq 0 \qquad (28).$$

3 TENZOR NAPETOSTI V VZDOLŽNO PREREZANEM VOTLEM VALJU

3 STRESS STATE ON THE LONGITUDINALLY SPLIT HOLLOW CYLINDER

Z upoštevanjem izrazov za integracijske konstante (26) in (27) lahko komponente napetostnega tenzorja zapišemo v obliki:

The elements of the stress tensor using the constants of integration (26) and (27) can be written as follows:

$$\bar{\sigma}_r = \bar{\sigma}_r^{(0)} + A\,\phi_0(x) - B\,\phi_1(x)\cos\vartheta \qquad (29),$$

$$\bar{\sigma}_\vartheta = \bar{\sigma}_\vartheta^{(0)} + A\,\psi_0(x) + B\,\psi_1(x)\cos\vartheta \qquad (30),$$

$$\bar{\tau}_{r\vartheta} = -B\,\phi_1(x)\sin\vartheta \qquad (31),$$

pri čemer so:

where:

$$\bar{\sigma}_r^{(0)} = \frac{3+\nu}{8}\left[1+\lambda^2-\frac{\lambda^2}{x^2}-x^2\right] = \frac{3+\nu}{8}\,\frac{1}{x^2}(x^2-\lambda^2)(1-x^2) \qquad (32),$$

$$\bar{\sigma}_\vartheta^{(0)} = \frac{3+\nu}{8}\left[1+\lambda^2+\frac{\lambda^2}{x^2}-\frac{1+3\nu}{3+\nu}\,x^2\right] \qquad (33)$$

napetosti v neprerezanem votlem valju in:

which represent the stress tensor elements that appear in the hollow cylinder without the longitudinal split, and:

$$\phi_0(x) = -\frac{1}{x^2}\left[\lambda^2\ln(\lambda)\,(1-x^2) - (1-\lambda^2)\,x^2\ln(x)\right] \qquad (34),$$

$$\phi_1(x) = \left[\frac{\lambda^2}{x^3} - \frac{1+\lambda^2}{x} + x\right] = -\frac{1}{x^3}(x^2-\lambda^2)(1-x^2) \qquad (35),$$

$$\psi_0(x) = \left[\lambda^2\ln\lambda\left(1+\frac{1}{x^2}\right) + (1-\lambda^2)(\ln x + 1)\right] \qquad (36),$$

$$\psi_1(x) = \left[\frac{\lambda^2}{x^3} + \frac{1+\lambda^2}{x} - 3x\right] \qquad (37)$$

nove funkcije.

additional functions:

4 NOTRANJI SILI IN MOMENTI

4 INTERNAL FORCES AND MOMENT

Z uporabo enačb za izračun napetosti (29), (30) in (31) lahko izračunamo normalno silo $\bar{N}$ in prečno silo $\bar{Q}$ ter upogibni moment $\bar{M}_0$ na poljubnem radialnem prerezu valja. Z upoštevanjem, da je:

Using the known functions of stress tensor (29), (30) and (31) we can now calculate the internal normal force $\bar{N}$, internal transversal force $\bar{Q}$ and internal bending moment $\bar{M}_0$. Considering that:

$$\int_\lambda^1 \psi_0(x)\,dx = \int_\lambda^1 \psi_1(x)\,x\,dx = 0$$

in da sta sila in upogibni moment, ki ju povzroča napetost $\sigma_\vartheta^{(0)}$, enaki:

and that the force and the bending moment induced by the stress $\sigma_\vartheta^{(0)}$, are equal:

$$\bar{F}^{(0)} = \int_\lambda^1 \bar{\sigma}_\vartheta^{(0)}\,dx = \frac{1}{3}(1-\lambda^3) \qquad (38),$$

$$\bar{M}_0^{(0)} = \int_\lambda^1 \bar{\sigma}_\vartheta^{(0)}\,x\,dx = \frac{3+\nu}{8}\left[\frac{5-\nu}{4(3+\nu)}(1-\lambda^4) - \lambda^2\ln\lambda\right] \qquad (39)$$

dobimo:

we get:

$$\overline{N} = \int_{\lambda}^{1} \overline{\sigma}_{\vartheta}\, dx = \overline{P}^{(0)} - B\,[1 - \lambda^2 + (1+\lambda^2)\ln\lambda]\cos\vartheta \tag{40}$$

$$\overline{Q} = \int_{\lambda}^{1} \overline{\tau}_{r\vartheta}\, dx = B\,[1 - \lambda^2 + (1+\lambda^2)\ln\lambda]\sin\vartheta \tag{41}$$

$$\overline{M}_0 = \int_{\lambda}^{1} \overline{\sigma}_{\vartheta}\, x\, dx = \overline{M}_0^{(0)} + [(1-\lambda^2)^2 - (2\lambda\ln\lambda)^2]\,\frac{A}{4} \tag{42}$$

5 IZPOLNITEV ROBNEGA POGOJA NA PREREZNIH PLOSKVAH

Kakor je bilo že omenjeno, se robnega pogoja (10) ne da natančno izpolniti. Konstanti A in B zato skušamo določiti tako, da bo dobljena rešitev najboljša aproksimacija robnega pogoja (10). V ta namen uporabimo metodo uravnoteženih ostankov, po kateri konstanti A in B izračunamo iz pogoja:

5 FULFILLMENT OF THE BOUNDARY CONDITION ON THE LONGITUDINAL SPLIT

As mentioned earlier, the condition (10) cannot be fully satisfied. We will therefore try to determine the constants A and B so that the obtained solution will be the best possible approximation of the boundary condition (10). To do this we chose the method of equilibrium remainder and tried to define A and B so that:

$$\int_{\lambda}^{1} \overline{\sigma}_{\vartheta}(x,\pi;A,B)\,\chi_n(x)\, dx = 0 \quad (n=1,2) \tag{43}$$

kjer sta $\varkappa_n(x)$ t.i. utežni funkciji. Glede na njuno izbiro imamo več metod, od katerih jih bo nekaj uporabljenih v nadaljevanju.

where $\varkappa_n(x)$ so-called weight functions. Considering the choice of weight functions, it is possible to use several methods.

5.1 **Metoda momentov (Metoda I)**

Fizikalno najbližji pogoj zahtevi (10) je, da napetost $\overline{\sigma}_v$ na prereznih ploskvah ne povzroča normalne sile in upogibnega momenta. V tem primeru pogoj (10) nadomestimo s pogojema:

5.1 **Method of Moments (Method I)**

Physically, the closest condition meeting the requirement (10) is that stress on the longitudinal section planes of the hollow cylinder does not generate a normal force and bending moment. In this case, requirement (10) can be replaced by the conditions:

$$\overline{N} = \int_{\lambda}^{1} \overline{\sigma}_{\vartheta}\, dx = 0 \quad \text{in} \quad \overline{M}_0 = \int_{\lambda}^{1} \overline{\sigma}_{\vartheta}\, x\, dx = 0 \quad \text{za} \ \vartheta = \pi \tag{44}$$

Iz (40) in (42) dobimo neznani konstanti:

From (40) and (42) we get the unknown constants:

$$A = -\frac{4\,\overline{M}^{(0)}}{(1-\lambda^2)^2 - (2\lambda\ln\lambda)^2} \tag{45}$$

$$B = -\frac{\overline{P}^{(0)}}{1 - \lambda^2 + (1+\lambda^2)\ln\lambda} \tag{46}$$

V poljubnem prerezu valja določimo po enačbah od (41) do (42) notranji sili in upogibni moment:

Thus, considering the above values of constants, we get in any section of the hollow cylinder using the equations (41) to (42), the following internal forces and internal bending moment:

$$\overline{N} = \overline{P}^{(0)}(1 + \cos\vartheta), \quad \overline{Q} = -\overline{P}^{(0)}\sin\vartheta, \quad \overline{M}_0 = 0 \tag{47}$$

Če se upogibni moment preoblikuje na nevtralno os x_n, ki je v tem primeru enaka [3]:

The natural axis of the curved beam is, [3]:

$$x_n = \frac{\lambda - 1}{\ln\lambda} \tag{48}$$

sledi izraz za upogibni moment:

The bending moment is now:

$$\bar{M} = \int_{(\eta)} \bar{\sigma}_{\vartheta}\, \xi\, d\xi = \int_{\lambda}^{1} \bar{\sigma}_{\vartheta}\, x\, dx - x_n \int_{\lambda}^{1} \bar{\sigma}_{\vartheta}\, dx = -x_n \bar{P}^{(0)}(1 + \cos\vartheta) \qquad (49),$$

ki se ujema z izrazom za upogibni moment, določenim po teoriji ukrivljenih nosilcev [1].

which is the same as the bending moment that was obtained using the formula of the curved beams theory [1].

5.2 Metoda najmanjših kvadratov (Metoda II)

Po tej metodi sledi pogoj za določitev integracijskih konstant na podlagi zahteve, da je kvadrat odstopanja približne rešitve od prave minimalen:

5.2 The Least Square Method (Method II)

Following this method, the condition for determining the constants of integration based on the requirement that the square of the deviation of the approximate solution from the real one has minimum is:

$$I(A,B) = \int_{\lambda}^{1} [\bar{\sigma}_{\vartheta}(x,\pi)]^2 x\, dx = min \qquad (50),$$

torej mora biti:

thus, it follows that:

$$\frac{\partial I}{\partial A} = 0 \text{ in } \frac{\partial I}{\partial B} = 0 \qquad (51).$$

Z izvedbo nakazanih operacij dobimo sistem enačb:

Performing the suggested operations, we get the following system of equations:

$$\begin{aligned} \left(\int_{\lambda}^{1} \psi_0^2\, dx\right) A - \left(\int_{\lambda}^{1} \psi_0 \psi_1\, dx\right) B &= -\left(\int_{\lambda}^{1} \bar{\sigma}_{\vartheta}^{(0)} \psi_0\, dx\right) \\ \left(\int_{\lambda}^{1} \psi_0 \psi_1\, dx\right) A - \left(\int_{\lambda}^{1} \psi_1^2\, dx\right) B &= -\left(\int_{\lambda}^{1} \bar{\sigma}_{\vartheta}^{(0)} \psi_0\, dx\right) \end{aligned} \qquad (52).$$

Vse integrale, ki se pojavljajo v zgornjem sistemu, je mogoče elementarno rešiti, vendar so dobljeni izrazi izredno obsežni, zato jih v tem prispevku ne navajamo.

All the integrals appearing in the above system can be solved, however the values obtained are too long, and are therefore not quoted here.

5.3 Kolokacijska metoda (Metoda III)

Ker sta na voljo dve integracijski konstanti, lahko izpolnimo pogoj (10) natančno v dveh točkah. Če sta to točki $x = \lambda$ in $x = 1$, lahko zapišemo:

5.3 Collocation Method (Method III)

As we have two constants of integration available, we can meet the condition (10) at two points. Choosing points $x = \lambda$ and $x = 1$, we get:

$$\bar{\sigma}_{\vartheta}(\lambda,\pi) = \bar{\sigma}_{\vartheta}(1,\pi) = 0 \qquad (53)$$

oziroma z upoštevanjem (31) sistem dveh enačb z dvema neznankama:

or, considering equation (31) we get a system of two equations with two unknowns:

$$\begin{aligned} (2\ln\lambda - \lambda^2 + 1)\, A - 2\,\frac{1-\lambda^2}{\lambda}\, B &= -\frac{1}{4}\,[(3+\nu^*) - (1-\nu^*)\lambda^2] \\ (2\lambda^2 \ln\lambda - \lambda^2 + 1)\, A + 2(1-\lambda^2)\, B &= -\frac{1}{4}\,[1-\nu^* - (3+\nu^*)\lambda^2] \end{aligned} \qquad (54),$$

katerega rešitev je:

whose solution is:

$$A = -\frac{(1-\nu^*)(1+\lambda^2) + 2(1+\nu^*)\lambda}{4[1-\lambda^2 + 2\lambda\ln\lambda]} \qquad (55),$$

$$B = \frac{\lambda[(1+\nu^*)(1-\lambda^2) - (1-\nu^*)(1+\lambda^2)\ln\lambda]}{4(1-\lambda^2 + 2\lambda\ln\lambda)(1+\lambda)} \qquad (56).$$

5.4 Mešana metoda (Metoda IV)

S predpostavko, da je upogibni moment na prereznih ploskvah enak nič, ima konstanta A vrednost, podano s (45), ne glede na vrednost konstante B. Iz pogoja:

5.4 Combined Method (Method IV)

Assuming that the bending moment on the longitudinal split planes of the hollow cylinder is equal to zero, constant A has the value defined by (45) irrespective of the value of constant B. From the conditions:

$$\bar{\sigma}_{\vartheta}(\lambda,\pi) = 0 \tag{57}$$

in enačbe (30) izhaja vrednost druge konstante:

using the equations (30), we then get:

$$B = \frac{\lambda}{1-\lambda^2}\left[\frac{1}{8}\left[(3+\nu^*)-(1-\nu^*)\lambda^2\right] - \frac{2\,(2\ln\lambda - \lambda^2 + 1)}{(1-\lambda^2)^2\,(2\lambda\ln\lambda)^2}\,\bar{M}_0^{(0)}\right] \tag{58}$$

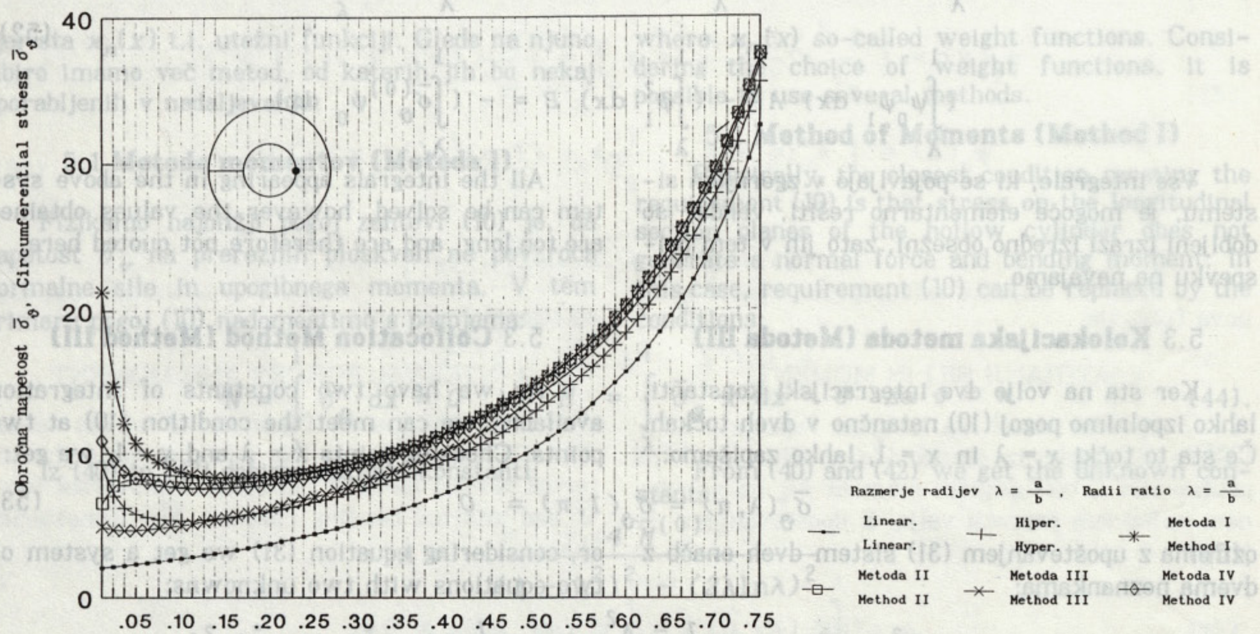
6 NUMERIČNI IZRAČUN

V preglednici 1 in na sliki 2 so podane vrednosti obročne napetosti pri kotu $\vartheta = 0$ na notranjem robu valja, pri čemer je izračun izveden z vrednostjo $\nu = 0{,}3$. V preglednico sta vključena tudi izračun napetosti s predpostavko linearne in hiperbolične porazdelitve upogibnih napetosti, ki jih dobimo z enačbami osnovne trdnosti [3] in upogibnega momenta, podanega z enačbo (42). V primeru linearne porazdelitve predpostavimo, da se nevtralna os ujema s težiščno osjo x_0, ki je podana z zvezo:

6 NUMERICAL CALCULATION

Table 1 presents the values of the circumferential stress at the angle $\vartheta = 0$ on the inner edge of the ring. The table includes also the calculations of stresses under the assumption of linear and hyperbolic distributions of the bending stress which are obtained by means of elementary strength equations [3] and the bending moment defined by equation (42). In the case of a linear distribution, we assume that the neutral axis coincides with the centre-of-gravity axis x_0, which is calculated from:

$$x_0 = \frac{1+\lambda}{2} \tag{59}$$



Sl. 2. *Obročne napetosti pri kotu $\vartheta = 0$ na notranjem robu valja*

Fig. 2. *The values of the circumferential stress at the angle $\vartheta = 0$ on the inner edge of the cylinder*

Upogibni moment je potem iz (49) enak:

The bending moment is, by equation (49):

$$\bar{M} = \frac{(1-\lambda^3)(1+\lambda)}{6}\,(1+\cos\vartheta) \tag{60}$$

Pri $\vartheta = 0$ je moment največji. Torej so na tem mestu največje tudi napetosti, ki so zaradi pravokotnega prereza valja podane z enačbo:

At $\vartheta = 0$ we get the maximal value of the bending moment and thus also the stresses which are, considering the rectangular split:

$$\bar{\sigma}_{\vartheta\max} = 2\,\frac{(1-\lambda^3)(1+\lambda)}{(1-\lambda)^2} \tag{61}$$

V primeru hiperbolične porazdelitve pa napetosti izračunamo z enačbo:

In the case of hyperbolic distribution, we can calculate the stresses by the equation:

$$\bar{\sigma}_{\vartheta} = \frac{\bar{M}}{1-\lambda}\,\frac{x - x_n}{(x_0 - x_n)\,x} \qquad (62),$$

iz katere lahko določimo za $\vartheta = 0$ in $x = \lambda$, največje obročne napetosti:

from which we get for $\vartheta = 0$ and $x = \lambda$ the maximum circumferential stresses:

$$\bar{\sigma}_{\vartheta max} = \frac{2}{3}\,\frac{1-\lambda^3}{1-\lambda}\,\frac{x_n(\lambda - x_n)}{(x_0 - x_n)\,\lambda} \qquad (63).$$

Kakor je razvidno iz preglednice 1, se velikosti napetosti po vseh metodah do $\lambda \cong 0{,}2$ manjšajo in se, razen pri linearni porazdelitvi, nato večajo, in to najbolj po izračunu z metodo I in najmanj po metodi III. Podobni rezultati so objavljeni v [1]. V omenjenem članku so vrednosti rezultatov izračuna po teoriji ukrivljenih nosilcev nekoliko višje od vrednosti rezultatov, dobljenih s hiperboličnim približkom, ker so avtorji uporabili za izračun momenta enačbo (60) in s tem zanemarili premaknitev nevtralne osi iz težišča.

As can be seen from Table 1, the values of stresses calculated by all four methods decrease until $\lambda \cong 0.2$, and then increase, except at linear distribution, the increase being highest by Method I and lowest by Method III. Similar results were published in [1]. In the paper, the results of the calculation according to the theory of curved beams are somewhat higher that those obtained by hyperbolic approximation. This difference appeared because the bending moment was calculated to the center of gravity and not to the neutral axis.

Preglednica 1. *Normalizirana obročna napetost σ_ϑ na notranjem robu pri $\vartheta = 0$*
Table 1. *Normalized stresses σ_ϑ at the inner for $\vartheta = 0$*

λ	lin/lin	hiper/hyper	I	II	III	IV
0,0125	2,077	9,223	21,215	6,431	4,845	10,905
0,0250	2,156	6,970	14,635	7,681	4,646	9,469
0,0500	2,327	5,718	10,765	8,185	4,627	8,269
0,1000	2,713	5,287	8,707	7,967	4,955	7,541
0,2000	3,720	5,930	8,365	8,175	6,121	7,835
0,3000	5,163	7,368	9,456	9,365	7,841	9,131
0,4000	7,280	9,601	11,536	11,479	10,299	11,314
0,5000	10,500	13,003	14,873	14,832	13,909	14,715
0,6000	15,680	18,412	20,265	20,235	19,524	20,153
0,7000	24,820	27,819	29,685	29,639	29,141	29,609
0,8000	43,920	47,221	49,119	50,481	48,760	49,073
0,9000	102,979	106,621	108,558	–	108,379	108,537
0,9500	222,492	226,343	228,272	–	228,187	228,257
0,9875	942,075	945,391	948,235	–	948,024	948,421

Kakor vidimo iz preglednice 1, je predpostavka linearne porazdelitve uporabna le za tanke votle valje z $\lambda \geqq 0{,}8$, hiperbolična pa za $\lambda \geqq 0{,}575$. Metodi II in IV dajeta praktično enake rezultate kakor metoda I za $\lambda \geqq 0{,}14$, vrednosti izračuna z metodo III pa so nekoliko nižje.

Preglednica 2 podaja največje napetosti, ki se pojavljajo na prereznih ploskvah in pomenijo napako modela. Po metodah I in II je to na notranjem robu votlega valja, pri preostalih dveh pa nekje znotraj prerezne ploskve. Pri vseh štirih metodah je napaka praktično zanemarljiva v območju od $\lambda \geqq 0{,}3$ do $\lambda = 0{,}9$, za manjše vrednosti λ pa napetosti, izračunane z metodama I in II, niso zanemarljive v primerjavi z največjo napetostjo v valju, ki je podana v preglednici 1. Vrednosti napetosti v preglednici 1 so zato za nižje velikosti λ vprašljive. Med vsemi metodami daje v vseh primerih metoda III najmanjšo napetost na prerezni ploskvi, in sicer:

The linear contribution of circumferential stress is acceptable only for hollow cylinders with a relatively thin wall, $\lambda \geqq 0.8$. In the case when the circumferential stress is hyperbolically distributed, the domain is a little bit larger, $\lambda \geqq 0.575$. In the domain $\lambda \geqq 0.14$ Methods I, II, and IV give almost the same value of the circumferential stress, but by Method III its values are a little lower.

Table 2 shows the maximum circular stresses occurring on the longitudinal split of the hollow cylinder which present the error of the individual methods. According to Methods I and II, the maximum circular stress occurs on the inner edge of the longitudinal cross section, and by the other two methods somewhere inside the same cross-section. According to all four methods, the error incurred is practically negligible in the domain $\lambda \geqq 0.3$ to $\lambda = 0.9$, for smaller values λ, however, the values of stresses on the split surfaces, obtained by Methods I and II, are not negligible compared to the maximum circumferential stress in table I. From all the methods studied, method III yields in all the cases the lowest circumferential stress on the edge of the split surface, i.e.

(64),

vendar pa po tej metodi dobimo na prerezni ploskvi normalno silo in upogibni moment, ki nista enaka nič. Nekoliko večjo napetost na prerezni ploskvi daje metoda IV, vendar je v tem primeru upogibni moment na tem območju enak 0.

However, using this method, the normal force and bending moment are obtained on the split surfaces not being equal to zero. A somewhat higher stress on the split surfaces is obtained by Method IV, nevertheless, in this case, the bending moment on the split surfaces is equal to zero.

Preglednica 2. *Največja obročna napetost σ_ϑ na prerezni ploskvi*
Table 2. *Maximum stress on the longitudinal split of the hollow cylinder*

λ	I	II	III	IV
0,0125	-10,310	-1,866	0,585	1,106
0,0250	-5,165	-1,646	0,503	0,841
0,0500	-2,495	-1,226	0,424	0,610
0,1000	-1,165	-0,762	0,346	0,421
0,2000	-0,529	-0,418	0,263	0,273
0,3000	-0,325	-0,279	0,208	0,202
0,4000	-0,222	-0,201	0,165	0,155
0,5000	-0,157	-0,147	0,129	0,120
0,6000	-0,111	-0,106	0,097	0,090
0,7000	-0,075	-0,074	0,068	0,065
0,8000	-0,046	-	0,043	0,041
0,9000	-0,021	-	0,020	0,020
0,9875	0,587	-	0,027	0,343

7 SKLEP

V tej razpravi je pokazano, da rešitev problema določitve napetostnega stanja v vzdolžno prerezanem valju v okviru splošne rešitve biharmonične enačbe s trigonometrijsko vrsto (11) nima točne rešitve, ker ni mogoče natančno izpolniti robnega pogoja (10) na prereznih ploskvah. Z numeričnim izračunom je bilo dokazano, da je običajna metoda izpolnjevanja robnega pogoja na prereznih ploskvah (metoda I) praktično uporabna za $\lambda > 0,2$. Za manjše vrednosti postane napaka izpolnjevanja robnega pogoja na prereznih ploskvah prevelika. Druge prikazane metode izpolnjevanja robnega pogoja na prereznih ploskvah prav tako ne dajejo natančnega odgovora na vprašanje, kakšno je napetostno stanje v primeru $\lambda \to 0$. Natančnejši odgovor na to vprašanje je verjetno v formulaciji naloge z metodami analitičnih funkcij.

7 CONCLUSION

From the results, it can be concluded that the exact solution of the stress state in a hollow cylinder which is split longitudinally along the rotation axis is impossible. The reason is that, when the general solution of biharmonic differential is an equation expressed with the trigonometric sum (11), the boundary conditions (10) on the split planes cannot be fulfilled exactly. The numerical method approach proved that method I can be used in domain $\lambda > 0.2$. In domain $\lambda < 0.2$, the error in fulfilling the boundary condition on the split planes becomes too large. The other three methods for fulfilling the boundary conditions on the split planes do not answer the question of what is the exact function of the stress state in the case $\lambda \to 0$. A better answer to this question can probably be obtained from results which will be determined by using analytical function in a complex plane.

8 LITERATURA
8 REFERENCES

[1] Bandara, C.–Nicolich, M.–Strozzi, A.: On the Bursting Mechanism in Rotating Rings; J. Strain Analysis, 28, 1993/153–162.

[2] Prelog, E.: Mehanika konstrukcij, 1994, Ljubljana

[3] Škerlj, M.: Mehanika, Trdnost, 1971, Ljubljana

Naslov avtorjev: mag. Milan Batista, dipl. inž.,
Visoka pomorska in prometna šola, Piran
prof. dr. Franc Kosel, dipl. inž.
Fakulteta za strojništvo
Univerze v Ljubljani
Ljubljana, Aškerčeva 6

Authors' Address: Mag. Milan Batista, Dip Ing.
Maritime and Trafic College, Piran
Prof. Dr. Franc Kosel, Dipl. Ing.
Faculty of Mechanical Engineering
University of Ljubljana
Ljubljana, Aškerčeva 6, Slovenia

Prejeto: 17.5.1994
Sprejeto:

Sprejeto:
Accepted: 23.2.1995