© Strojniški vestnik (43) št. 5-6, str. 219-230, 1997
Tiskano v Sloveniji. Vse pravice pridržane.
UDK 621.373:519.61/.64:534.1/.2

SV:0039-2480(97)3-R4

© Journal of Mechanical Engineering (43) No. 5-6, pp. 219-230, 1997
Printed in Slovenia. All rights reserved.
UDC 621.373:519.61/.64:534.1/.2

Analiza nelinearnih oscilatorjev z več prostostnimi stopnjami
Analysis of Nonlinear Oscilators with Finite Degrees of Freedom

RUDI PUŠENJAK

V članku je prikazana analiza nelinearnih oscilatorjev z več prostostnimi stopnjami, ki lahko izvajajo lastna nihanja (avtonomni sistemi), ali jih vzbujamo z zunanjimi periodičnimi signali. Za izračun ustaljenega odziva v časovnem prostoru, za izračun limitnih ciklov, konstrukcijo stabilnih in nestabilnih vej resonančnih krivulj ter za določitev diagramov rešitev v odvisnosti od parametrov nelinearnega dinamičnega sistema je uporabljena metoda koračnega harmonskega ravnovesja. Z uporabo Floquetove teorije določimo stabilnost izračunanih rešitev. V zgledih je z uporabo metode KHR izračunana primarna resonanca, superharmonična resonanca in subharmonična resonanca Duffingovega oscilatorja. Med avtonomnimi sistemi je obdelan Van der Polov oscilator z različnimi stopnjami nelinearnosti in sklopljeni nelinearni oscilator z dvema prostostnima stopnjama s prikazom diagrama rešitve. Izračunane so resonančne krivulje tega sistema ob spremenljivi frekvenci zunanjega vzbujevalnega signala.

Ključne besede: nelinearni oscilatorji, metoda koračnega harmonskega ravnovesja, resonanca primarna, resonanca superharmonična, resonanca subharmonična

This paper presents the analysis of nonlinear oscillators with finite degrees of freedom exhibiting self-sustained oscillations (autonomous systems) or performing oscillations under periodic excitation. The incremental harmonic balance method is used for calculation of the steady time response, limit cycles, stable and unstable branches of the resonance curves and for construction of solution diagrams in dependence of system parameters. The stability of the calculated solutions is proved by Floquet theory. In the paper construction of primary, superharmonic and subharmonic resonances of higher order of the Duffing oscillator is made by the use of IHB method. Autonomous systems are studied on the examples of the Van der Pol oscillator and coupled nonlinear oscillator with two degrees of freedom, where the solution diagram is shown. Additionally, the resonance curves for the same nonlinear oscillator under excitation with variable frequency are calculated and plotted.

Keywords: nonlinear oscillators, incremental harmonic balance method, primary reonance, superharmonic reonance, subharmonic resonance

0 UVOD

Raziskave nihanj nelinearnih oscilatorjev kažejo, da so analitične rešitve teh nihanj mogoče samo v redkih, izjemnih primerih. Če se omejimo na avtonomne nelinearne oscilatorje in nelinearne oscilatorje, ki jih vzbujamo s periodičnimi signali, ugotovimo, da so odzivna nihanja po preteku prehodnega pojava periodična samo v določenem območju parametrov in da obstajajo razvejitvene točke (bifurkacije) v parametričnem prostoru, ki delijo območja kakovostno povsem različnih rešitev. Tako ločimo območja stabilnih periodičnih nihanj od nestabilnih, ki s časom neomejeno naraščajo in fizikalno niso mogoča. Poseben primer se pojavi, kadar so vsa možna periodična nihanja nestabilna, vendar ostajajo omejena: pravimo, da se sistem obnaša kaotično. V tem članku bomo periodične rešitve poiskali z uporabo metode koračnega harmonskega ravnovesja (KHR), čeprav je mogoče uporabiti tudi metode numerične integracije in različne perturbacijske metode. Metode numerične integracije računalniško niso učinkovite, ker nas zanima ustaljeni pojav, ki ga lahko izračunamo šele potem, ko prehodni pojav izzveni. Perturbacijske metode [1] so učinkovite in dodatno omogočajo globalno analizo stabilnosti, vendar so omejene na dinamične sisteme z majhno stopnjo nelinearnosti.

0 INTRODUCTION

Investigations of oscillations in nonlinear oscillators reveal that analytical solutions exist in rare, exceptional cases. If we restrict the study to autonomous nonlinear oscillators and periodically excited nonlinear oscillators, we find that periodic responses after the transient period are observed only in certain range of parameters and therefore bifurcation points exist in parametric space, and are separating the domains of qualitively different solutions. Moreover, ranges of stable periodic oscillations can be distinguished from unstable periodic solutions, which cannot physically exist. A special example occurs, when all possible periodic oscillations are unstable but remain bounded: we say that system behaves chaotically. In this paper we will find periodic solutions by using the incremental harmonic balance method (IHB), although numerical time-integration methods and various perturbation methods can also be used. Methods of numerical integration cannot be computationally efficient, because they are looking for steady-state solutions which can be computed after transient period. Perturbation methods [1] are efficient and make global stability analysis possible, but they are limited to dynamical systems with small nonlinearity.

Z uporabo metode KHR frekvenčni potek resonančne krivulje periodično vzbujanih nelinearnih dinamičnih sistemov zlahka izračunamo tako, da v korakih povečujemo (inkrementiramo) frekvenco vzbujevalnega signala. Pri tem načeloma ni nobenih ovir, ne glede na to, ali gre za izračun stabilnih ali nestabilnih vej resonančne krivulje. S procesom priraščanja, v katerem smemo poljubno spreminjati nek nadaljnji sistemski parameter, lahko konstruiramo celotni diagram rešitev in v njem določimo točke razvejitev (bifurkacijske točke). Tako pri procesu večanja frekvence kakor pri procesu priraščanja izbranega sistemskega parametra izhajamo iz poprejšnje oziroma predpostavljene rešitve, nakar uvedemo Newton-Raphsonov iterativni postopek, s katerim izračunamo konvergentno rešitev, ki zadošča vnaprej določenemu tolerančnemu kriteriju.

By using the IHB method the frequency response of the resonance curve of periodically excited nonlinear oscillators can be easilly computed by incrementing the frequency of the exciting signal. In principle stable as well as unstable branches of the resonance curve can be computed without any trouble. If an additional parameter is varied inside the augmentation process, the complete solution diagram with bifurcation points is constructed. In the process of frequency incrementation as well as in the augmentation of the selected system parameter we proceed from the previous or assumed solution and then introduce the Newton-Raphson iterative procedure until a convergent solution is obtained subject to a given tolerance criterion..

1 METODA KORAČNEGA HARMONSKEGA RAVNOVESJA (KHR)

Nihanja nelinearnih dinamičnih sistemov z N prostostnimi stopnjami, ki so izpostavljeni periodičnemu vzbujanju, lahko opišemo s sistemom nelinearnih navadnih diferencialnih enačb:

1 INCREMENTAL HARMONIC BALANCE (IHB) METHOD

Oscillations of nonlinear dynamical systems with N degrees of freedom, subjected to periodic excitation can be described by a system of nonlinear ordinary differential equations:

$$\omega^2 \ddot{x}_i + f_i(x_1,\dots,x_N;\dot{x}_1,\dots,\dot{x}_N;\omega,\lambda) = q_i(\tau),\quad (i=1,\dots,N) \tag{1}$$

kjer uvedemo brezdimenzijski čas $\tau = \omega t$ kot neodvisno spremenljivko, s $f_i(x_1,\dots,x_N;\dot{x}_1,\dots,\dot{x}_N;\omega,\lambda)$ pa označimo nelinearne funkcije odvisnih spremenljivk x_i in pripadajočih prvih odvodov $\dot{x}_i = 1/\omega \,.\, \mathrm{d}x_i/\mathrm{d}t$, pri čemer je ω vzbujevalna frekvenca, λ pa pomeni nek prosti parameter. Predpostavimo, da vzbujamo oscilator s harmoničnim signalom:

where dimensionless time $\tau = \omega t$ is introduced as an independent variable and where $f_i(x_1,\dots,x_N;\dot{x}_1,\dots,\dot{x}_N;\omega,\lambda)$ denote nonlinear functions of dependent variables x_i and corresponding first derivatives $\dot{x}_i = 1/\omega \,.\, \mathrm{d}x_i/\mathrm{d}t$ with exciting frequency ω and arbitrary changeable parameter λ. Suppose, that the nonlinear oscillator is harmonically excited:

$$q_i(\tau) = g_i \cos[(2m-1)\tau]; \tag{2}$$

pri čemer pomeni m neko primerno celo število $m=1,2,3\dots$ Vzemimo neko začetno rešitev $x_{i0},\dot{x}_{i0},\omega_0,\lambda_0,g_{i_0};(i=1,\dots,N)$ ter skušajmo z Newton-Raphsonovim iterativnim procesom najti sosednjo rešitev:

where m is an appropriate integer $m=1,2,3,\dots$ Furthermore, let us assume an initial solution $x_{i0},\dot{x}_{i0},\omega_0,\lambda_0,g_{i_0};(i=1,\dots,N)$ and attempting to get the neighbouring solution by using iterative Newton-Raphson procedure :

$$x_i = x_{i_0} + \Delta x_i, \dot{x}_i = \dot{x}_{i_0} + \Delta \dot{x}_i, \omega = \omega_0 + \Delta\omega, \lambda = \lambda_0 + \Delta\lambda;$$
$$g_i = g_{i_0} + \Delta g_i; (i=1,\dots,N) \tag{3}$$

tako, da k začetni rešitvi dodajamo: $\Delta x_i, \Delta\dot{x}_i, \Delta\omega, \Delta\lambda, \Delta g_i$. Če izraza (3) vstavimo v enačbo (1) in enačbo (1) razvijemo v Taylorjeve vrste v okolici začetne rešitve, dobimo linearizirane koračne diferencialne enačbe:

with adding increments $\Delta x_i, \Delta\dot{x}_i, \Delta\omega, \Delta\lambda, \Delta g_i$ to the initial solution. Putting expressions (3) into equations (1) and developing equations (1) in Taylor's series about initial solution, linearized incremental differential equations are obtained:

$$\omega_0^2\Delta\ddot{x}_i+\sum_{j=1}^{N}\left(\frac{\partial f_i}{\partial \dot{x}_j}\right)_0\Delta\dot{x}_i+\sum_{j=1}^{N}\left(\frac{\partial f_i}{\partial x_j}\right)_0\Delta x_i=$$

$$=\left[g_{i_0}\cos[(2m-1)\tau]-\omega_0^2\ddot{x}_{i_0}-f_{i_0}\right]-\left[\left(\frac{\partial f_i}{\partial\omega}\right)_0+2\omega_0\ddot{x}_{i_0}\right]\Delta\omega-\left(\frac{\partial f_i}{\partial\lambda}\right)_0\Delta\lambda+\cos[(2m-1)\tau]\cdot\Delta g_i \quad (4),$$

pri čemer zanemarimo vse koračne izraze višjih redov. Če začetne vrednosti in pripadajoče prirastke sestavimo v naslednje vektorje:

where all incremental expressions of higher order are neglected. If initial values and corresponding increments are combined in the following vectors:

$$\{X_0\}=\left[x_{I_0},\dots,x_{N_0}\right]^T,\{\Delta X\}=\left[\Delta x_1,\dots,\Delta x_N\right]^T,\{G_0\}=\left[g_{I_0},\dots,g_{N_0}\right]^T,\{\Delta G\}=\left[\Delta g_1,\dots,\Delta g_N\right]^T,\{F_0\}=\left[f_{I_0},\dots,f_{N_0}\right]^T \quad (5)$$

in uvedemo matriki $[C]$ in $[K]$ z ustreznimi členi ter vektorje $\{Q\}$, $\{P\}$ in $\{R\}$ s pripadajočimi komponentami:

and matrices $[C]$ and $[K]$ as well as vectors $\{Q\},\{P\}$ and $\{R\}$ are defined with elements:

$$C_{ij}=\left(\frac{\partial f_i}{\partial \dot{x}_j}\right)_0, \qquad K_{ij}=\left(\frac{\partial f_i}{\partial x_j}\right)_0$$

$$Q_i=\left(\frac{\partial f_i}{\partial\omega}\right)_0,\ P_i=\left(\frac{\partial f_i}{\partial\lambda}\right)_0,\quad R_i=g_{i_0}\cos[(2m-1)\tau]-\left(\omega_0^2\ddot{x}_{i_0}+f_{i_0}\right) \quad (6),$$

lahko linearizirane koračne diferencialne enačbe zapišemo v zgoščeni matrični obliki:

then the linearized incremental differential equations can be written in compact matrix form as:

$$\omega_0^2\{\Delta\ddot{X}\}+[C]\cdot\{\Delta\dot{X}\}+[K]\cdot\{\Delta X\}=\{R\}-\left(2\omega_0\{\ddot{X}_0\}+\{Q\}\right)\Delta\omega-\{P\}\cdot\Delta\lambda+\cos[(2m-1)\tau]\cdot\{\Delta G\} \quad (7).$$

Dobljena matrična diferencialna enačba je linearna, vendar ima spremenljive koeficiente. Rešitev takega sistema enačb dobimo z Galerkinovim postopkom, ki pomeni drugi korak pri metodi KHR. Ker je vzbujanje periodično, lahko pričakujemo, da bo ustaljena rešitev prav tako periodična, razen če rešitev ni stabilna (npr. se s časom zvečuje) ali če se sistem obnaša kaotično. V primeru, da periodična ustaljena rešitev obstaja, moremo rešitev enačbe (7) opisati z nastavkom:

The matrix differential equation (7) is linear with periodic coefficients. The solution of this matrix equation can be obtained by the Galerkin procedure, which presents the second step in the IHB method. Because of periodic excitation, we can expect that the steady-state solution is also periodic, except in the case of unbounded solutions. If a periodic steady state solution exists, then the solution of equation (7) is expressed as follows:

$$x_{i_0}=\sum_{n=1}^{K}\left(a_{i,n}^0\cos[(2n-1)\tau]+b_{i,n}^0\sin[(2n-1)\tau]\right),$$

$$\Delta x_i=\sum_{n=1}^{K}\left(\Delta a_{i,n}\cos[(2n-1)\tau]+\Delta b_{i,n}\sin[(2n-1)\tau]\right);(i=1,\dots,N) \quad (8),$$

oziroma v matrični obliki:

or in matrix form:

$$x_{i_0}=[T]\{a_{i_0}\},\quad \Delta x_i=[T]\cdot\{\Delta a_i\},\quad (i=1,\dots,N) \quad (9),$$

kjer so:

where:

$$[T]=\left[\cos\tau,\cos 3\tau,\ldots,\cos[(2K-1)\tau];\sin\tau,\sin 3\tau,\ldots,\sin[(2K-1)\tau]\right] \tag{10a}$$

$$\left\{a_{i_0}\right\}=\left[a^0_{i,1},a^0_{i,2},\ldots,a^0_{i,K};b^0_{i,1},b^0_{i,2},\ldots,b^0_{I,K}\right]^T \tag{10b}$$

$$\left\{\Delta a_i\right\}=\left[\Delta a_{i,1},\Delta a_{i,2},\ldots,\Delta a_{i,K};\Delta b_{i,1},\Delta b_{i,2},\ldots,\Delta b_{i,K}\right]^T \tag{10c}.$$

Uvedemo še matriko $[Y]$ in vektorje $\{A_0\}$, $\{\Delta A\}$:

By introducing matrix $[Y]$ and vectors $\{A_0\},\{\Delta A\}$:

$$[Y]=\begin{bmatrix}[T] & [0] & \cdot & [0]\\ [0] & [T] & \cdot & [0]\\ \cdot & \cdot & \cdot & \cdot\\ [0] & [0] & \cdot & [T]\end{bmatrix},\quad \{A_0\}=\begin{bmatrix}\{a_{1_0}\}\\ \{a_{2_0}\}\\ \cdot\\ \{a_{N_0}\}\end{bmatrix},\{\Delta A\}=\begin{bmatrix}\{\Delta a_1\}\\ \{\Delta a_2\}\\ \cdot\\ \{\Delta a_N\}\end{bmatrix} \tag{11},$$

s katerimi lahko vektorje $\{X_0\}$, $\{\Delta X\}$ izrazimo v naslednji obliki:

the vectors $\{X_0\},\{\Delta X\}$ can be expressed as follows:

$$\{X_0\}=[Y]\cdot\{A_0\},\quad \{\Delta X\}=[Y]\cdot\{\Delta A\} \tag{12}.$$

Izvedimo Galerkinov postopek prek ene periode brezdimenzijskega časa $\tau=\omega t$, to je v mejah od 0 do 2π:

By performing the Galerkin procedure for one period of dimensionless time $\tau=\omega t$, that is on the interval 0 to 2π:

$$\frac{1}{\pi}\int_0^{2\pi}\delta\{\Delta A\}^T\left\{[Y]^T\left[\omega_0^2[\ddot Y]+[C][\dot Y]+[K][Y]\right]\{\Delta A\}\right\}d\tau=$$

$$\frac{1}{\pi}\int_0^{2\pi}\delta\{\Delta A\}^T\left\{[Y]^T\left[\{R\}-\left(\{Q\}+2\omega_0[\dot Y]\{A_0\}\right)\Delta\omega-\{P\}\Delta\lambda+\{\Delta G\}\cos[(2m-1)\tau]\right]\right\}d\tau \tag{13},$$

kjer je vektor variacij harmonskih prirastkov $\delta\{\Delta A\}^T$ na obeh straneh enačbe (13) poljuben, s čimer dobimo linearno matrično enačbo za neznani vektor harmonskih prirastkov $\{\Delta A\}$:

where $\delta\{\Delta A\}^T$ presents arbitrary variations of harmonic increments vector, one obtains the linear matrix equation for unknown harmonic increments vector $\{\Delta A\}$:

$$[k]\{\Delta A\}=\{r\}+\{q\}\Delta\omega+\{p\}\Delta\lambda+[s]\{\Delta G\}\qquad [s]=\frac{1}{\pi}\int_0^{2\pi}[Y]^T\cos[(2m-1)\tau]\,d\tau. \tag{14},$$

kjer sta [k] in [s] matriki, {r}, {q}, {p} pa naslednji vektorji:

where [k] and [s] are matrices, and $\{r\},\{q\},\{p\}$ are the following vectors:

$$[k]=\frac{1}{\pi}\int_0^{2\pi}[Y]^T\left[\omega_0^2[\ddot Y]+[C][\dot Y]+[K][Y]\right]d\tau, \tag{15},$$

$$\{r\}=\frac{1}{\pi}\int_0^{2\pi}\left[\omega_0^2[\dot Y]^T\cdot\dot Y\{A_0\}-[Y]^T\cdot\{F_0\}+[Y]^T\{G_0\}\cos[(2m-1)\tau]\right]d\tau \tag{16},$$

$$\{q\}=\frac{1}{\pi}\int_0^{2\pi}\left[2\omega_0[\dot Y]^T[\dot Y]\{A_0\}-[Y]^T\{Q\}\right]d\tau, \tag{17},$$

$$\{p\}=-\frac{1}{\pi}\int_0^{2\pi}[Y]^T\{P\}d\tau, \tag{18}.$$

Matrično enačbo (14) moramo razrešiti na vsakem koraku Newton-Raphsonovega iterativnega postopka, dokler ne najdemo konvergentne rešitve. Ob predpostavki, da so $\Delta\omega, \Delta\lambda, \{\Delta G\}$ enaki nič in da je $\{A\}$ točna rešitev, je rezidualni vektor $\{r\}$ enak nič in enačba (14) zagotavlja, da je tedaj tudi $\{\Delta A\}$ enak nič. Če je $\{A\}$ aproksimacija, izračunamo boljši približek $\{A\}+\{\Delta A\}$ z razrešitvijo enačbe (14). Postopek ponavljamo, dokler $\{r\}$ ne postane dovolj majhen. Če vrednosti parametrov $\Delta\omega, \Delta\lambda, \{\Delta G\}$ variirajo, vzamemo prejšnjo rešitev za aproksimacijo in poiščemo novo rešitev z Newton-Raphsonovim iterativnim postopkom. Izmenjava procesa priraščanja parametrov z iterativnim postopkom omogoča konstrukcijo resonančnih krivulj z naraščanjem frekvence ω v izbranem območju (frekvenčno večanje) ali s spreminjanjem izbrane komponente vektorja harmonskih koeficientov A_n (amplitudno večanje). Z izmenjavo procesa priraščanja parametra λ z iterativnim postopkom lahko zgradimo diagram rešitve.

Ko je ustaljena rešitev z uporabo metode KHR izračunana, nas običajno zanima stabilnost dobljene rešitve. Vzemimo, da je $\{X_0\}$ ustaljena rešitev enačbe (7), tako da je $\{R\}=\{0\}$. Če frekvence, amplitude vzbujevalnega signala in parametra λ ne spreminjamo, zadošča majhna sprememba rešitve $\{X\}=\{X_0\}+\{\Delta X\}$ perturbirani enačbi:

The matrix eq. (14) is solved at each step of the Newton-Raphson iterative procedure until convergent solution is obtained. With the assumption that $\Delta\omega, \Delta\lambda, \{\Delta G\}$ vanish and $\{A\}$ represents an exact solution, the residual vector $\{r\}$ vanishes and equation (14) guarantes, that $\{\Delta A\}$ vanishes. If $\{A\}$ is an approximation, then the better solution $\{A\}+\{\Delta A\}$ is computed by solving the equation (14). This procedure is repeated until $\{r\}$ becomes sufficiently small. If the values of parameters $\Delta\omega, \Delta\lambda, \{\Delta G\}$ are changed, the previous solution is taken as an approximation and a new solution is calculated by Newton-Raphson's iterative procedure. The exchange of augmentation process and Newton-Raphson's iterative procedure enables the construction of resonance curves by incrementing the frequency ω in the selected range (frequency incrementation) or by varying certain component A_n of the harmonic coefficients vector (amplitude incrementation). By exchanging augmentation process for parameter λ and Newton-Raphson procedure, the solution diagram can be constructed.

When the steady-state solution is computed by using IHB method, usually the stability of the obtained solution is of great interest. Supposing, that $\{X_0\}$ means a steady-state solution of equation (7), which implies that $\{R\}=\{0\}$. If the exciting frequency, as well as the amplitude of the exciting signal and value of parameter λ are not changed, small changes of solution $\{X\}=\{X_0\}+\{\Delta X\}$ will satisfy the perturbed equation:

$$\omega_0^2\{\Delta\ddot{X}\}+[C]\{\Delta\dot{X}\}+[K]\cdot\{\Delta X\}=\{0\} \tag{19}$$

ki je matrična navadna linearna diferencialna enačba s periodičnimi koeficienti v $[C]$ ali v $[K]$ ali celo v obeh. Enačbo (19) lahko prevedemo na sistem prvega reda z uvedbo kombiniranih vektorjev:

which is a matrix ordinary linear differential equation with periodic coefficients in $[C]$ or $[K]$ or both. Equation (19) can be presented by first order system introducing combined vectors:

$$\frac{d}{d\tau}\begin{Bmatrix}\Delta X\\ \Delta\dot{X}\end{Bmatrix}=-\frac{1}{\omega_0^2}\begin{bmatrix}[0] & [I]\\ [K] & [C]\end{bmatrix}\begin{Bmatrix}\Delta X\\ \Delta\dot{X}\end{Bmatrix},$$

$$\{Z\}=\begin{Bmatrix}\Delta X\\ \Delta\dot{X}\end{Bmatrix},\quad [A]=-\frac{1}{\omega_0^2}\begin{bmatrix}[0] & [I]\\ [K] & [C]\end{bmatrix}, \tag{20}$$

$$\{\dot{Z}\}=[A]\{Z\}$$

tako da ustreza stabilnost ustaljene rešitve stabilnosti rešitev sistema $\{\dot{Z}\}=[A]\{Z\}$. Naj bo $[\Phi(\tau)]$ prenosna matrika, s katero prevedemo vektor $\{Z(\tau)\}$ v času $\tau=0$ v vektor $\{Z(\tau+2\pi)\}$ ob koncu ene periode. V skladu s Floquetovo teorijo [2] so rešitve sistema (20) stabilne, če so moduli lastnih vrednosti prenosne matrike na koncu ene periode $[\Phi(2\pi)]$ manjši od enote, v nasprotnem primeru pa so rešitve nestabilne. V ta namen razdelimo vsako periodo 2π na N_k korakov (τ_{k-1},τ_k), kjer je $\Delta_k=\tau_k\ \tau_{k-1}$ dolžina posameznega koraka in v vsakem od teh korakov periodično matriko koeficientov $[A(\tau)]$ nadomestimo s konstantno matriko:

With this transformation, the stability of the steady-state solution corresponds to the stability of solutions of the system $\{\dot{Z}\}=[A]\{Z\}$. Assuming that $[\Phi(\tau)]$ denotes the transition matrix, which is used to transform the state vector $\{Z(\tau)\}$ at the time $\tau=0$ in the state vector $\{Z(\tau+2\pi)\}$ at the end of one period. In accordance with Floquet theory [2], solutions of the system (20) are stable, if all moduli of the eigenvalues of the transition matrix at the end of one period $[\Phi(2\pi)]$ are less than unity, otherwise the solutions are unstable. For this purpose, each period 2π is divided into N_k intervals (τ_{k-1},τ_k), where $\Delta_k=\tau_k-\tau_{k-1}$ means the length of the individual intervals. In each interval the periodic matrix of coefficients $[A(\tau)]$ is replaced by a constant matrix:

$$[A_k] = \frac{1}{\Delta_k} \int_{\tau_{k-1}}^{\tau_k} [A(\zeta)] d\zeta \quad (21).$$

Prehodno matriko na koncu ene periode dobimo z obrazcem:

The transition matrix at the end of one period is given by :

$$[\Phi(2\pi)] = \prod_{k=1}^{N_k} exp(\Delta_k [A_k]) \quad (22).$$

2 VAN DER POL-DUFFINGOV OSCILATOR

Opisano metodo bomo uporabili pri periodično vzbujanem sklopljenem nelinearnem oscilatorju z dvema prostostnima stopnjama, ki ga lahko opišemo z naslednjima dvema nelinearnima navadnima diferencialnima enačbama (N=2):

2 VAN DER POL-DUFFING OSCILLATOR

Above method will now be used at a periodically excited coupled nonlinear oscillator with two degrees of freedom, which can be described by using the following two nonlinear ordinary differential equations (N=2):

$$\omega^2 \ddot{x}_1 - \lambda(\varepsilon_1 - \mu_1 x_1^2)\omega \dot{x}_1 + \upsilon_1 x_1 + \alpha_{11} x_1^3 + \alpha_{12} x_1^2 x_2 + \alpha_{13} x_1 x_2^2 + \alpha_{14} x_2^3 = g_1 cos[(2m-1)\tau] \quad (23a),$$

$$\omega^2 \ddot{x}_2 - \lambda(\varepsilon_2 - \mu_2 x_2^2)\omega \dot{x}_2 + \upsilon_2 x_2 + \alpha_{21} x_1^3 + \alpha_{22} x_1^2 x_2 + \alpha_{23} x_1 x_2^2 + \alpha_{24} x_2^3 = 0 \quad (23b).$$

V posebnem primeru, ko so vrednosti parametrov $\alpha_{12}, \alpha_{13}, \alpha_{14}$ in $\alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{24}$ vse enake nič, se zgornji nelinearni sistem razklopi in razpade na dva ločena nelinearna oscilatorja z eno prostostno stopnjo:

In special cases, when all parameter values $\alpha_{12}, \alpha_{13}, \alpha_{14}$ and $\alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{24}$ all vanish, the above nonlinear system is decomposed into two distinct nonlinear oscillators with one degree of freedom:

$$\omega^2 \ddot{x}_1 - \lambda(\varepsilon_1 - \mu_1 x_1^2)\omega \dot{x}_1 + \upsilon_1 x_1 + \alpha_{11} x_1^3 = g_1 cos[(2m-1)\tau], \quad (24a),$$

$$\omega^2 \ddot{x}_2 - \lambda(\varepsilon_2 - \mu_2 x_2^2)\omega \dot{x}_2 + \upsilon_2 x_2 = 0 \quad (24b),$$

ki sta lahko nelinearna oscilatorja Van der Polovega ali Van der Pol-Duffingovega tipa. Če ima tudi parameter μ_1 v enačbi (24a) vrednost nič, dobimo oscilator Duffingovega tipa. Matriki $[k],[s]$ in vektorji $\{r\},\{q\},\{p\}$ za sklopljeni nelinearni oscilator z dvema prostostnima stopnjama, opisan z enačbama (23 a,b), so izpeljani v [3].

which can present nonlinear oscillators of the Van der Pol or Van der Pol-Duffing type. If the parameter value μ_1 in the equation (24a) is also zero, one obtains the Duffing oscillator. Matrices $[k],[s]$ and vectors $\{r\},\{q\},\{p\}$ for coupled nonlinear oscillator with two degrees of freedom, described by equations (23a,b), are derived in [3].

3 NUMERIČNI ZGLEDI

Duffingov oscilator

Če v enačbi (24a) izberemo vrednost parametra $\mu_1 = 0$, dobimo Duffingov oscilator:

3 NUMERICAL EXAMPLES

Duffing oscillator

With the value $\mu_1 = 0$ in the equation (24a) one obtains the Duffing oscillator:

$$\omega^2 \ddot{x}_1 - \lambda \varepsilon_1 \omega \dot{x}_1 + \upsilon_1 x_1 + \alpha_{11} x_1^3 = g_1 cos[(2m-1)\tau] \quad (25).$$

Pri določitvi resonančne krivulje Duffingovega oscilatorja se poleg stabilne veje pojavi tudi nestabilna veja, ki jo v tehničnem pomenu laže zgradimo, če zamenjamo proces frekvenčnega povečevanja s procesom fazne povečave. Fazne prirastke vpeljemo z uporabo nastavka:

When the resonance curve of the Duffing oscillator is constructed, stable as well as unstable branches appear. The complete resonance curve is easier to obtain, if the frequency incrementation process is exchanged for the phase incrementation process. Phase increments are introduced by using the following formula:

$$x_{i_0} = \sum_{n=1}^{K}\left(a^0_{i,n}\cos[(2n-1)\tau] + b^0_{i,n}\sin[(2n-1)\tau]\right) = \sum_{n=1}^{K} A_n \cos[(2n-1)\tau - \Phi_n] =$$

$$= \sum_{n=1}^{K}\left(A_n \cos\Phi_n \cos[(2n-1)\tau] + A_n \sin\Phi_n \sin[(2n-1)\tau]\right) \quad (26),$$

kjer so A_n velikosti, Φ_n pa fazni koti. Transformacije med harmonskimi amplitudami $a^0_{i,n}$, $b^0_{i,n}$ in velikostmi A_n ter faznimi koti Φ_n so določene z razmerji:

where A_n denotes magnitudes and Φ_n denotes phase angles. Transformations between harmonic amplitudes $a^0_{i,n}$, $b^0_{i,n}$ and magnitudes A_n and phase angles Φ_n are determined by relations:

$$a^0_{i,n} = A_n \cos\Phi_n, \qquad \left\{\begin{matrix}\Delta a_{i,n}\\ \Delta b_{i,n}\end{matrix}\right\} = \begin{bmatrix}\cos\Phi_n & -b^0_{i,n}\\ \sin\Phi_n & a^0_{i,n}\end{bmatrix}\left\{\begin{matrix}\Delta A_n\\ \Delta\Phi_n\end{matrix}\right\} \quad (27a),$$

$$b^0_{i,n} = A_n \sin\Phi_n, \quad (27b).$$

V naslednjem zgledu je po tej metodi izračunana primarna resonanca, superharmonična resonanca tretjega reda in subharmonična resonanca tretjega reda Duffingovega oscilatorja, pri čemer so vrednosti posameznih parametrov:

In the next example the phase incrementation method is used for computing primary resonance, superharmonic and subharmonic resonance of the third order of the Duffing oscillator, where parameter values:

$$\lambda = -2; \ \ \varepsilon_1 = 0{,}01; \ \ \upsilon_1 = 1; \ \ \mu_1 = 0; \ \ \alpha_{11} = 0{,}1$$

v vseh treh primerih, amplitude vzbujevalnega signala in harmonski indeksi m pri posameznih vrstah resonanc pa so:

$g_1 = 0{,}1; \ m = 1$ pri primarni resonanci,

$g_1 = 1{,}0; \ m = 1$ pri superharmonični resonanci tretjega reda in

$g_1 = 8{,}0; \ m = 2$ pri subharmonični resonanci tretjega reda.

are equal in all three cases, whereas amplitudes of exciting signal are chosen as:

$g_1 = 0{,}1; \ m = 1$ of primary resonance,

$g_1 = 1{,}0; \ m = 1$ for superharmonic resonance of the third order,

$g_1 = 8{,}0; \ m = 2$ for subharmonic resonance of the third order.

Preglednica 1 prikazuje rezultate primarne resonance, superharmonične resonance tretjega reda in subharmonične resonance tretjega reda Duffingovega oscilatorja, izračunane z metodo fazne povečave. Rezultati so preverjeni z izračuni, pri katerih je upoštevano različno, vendar zadostno število harmonikov. Čeprav se število iteracij pri različnem številu upoštevanih harmonikov na splošno razlikuje, dobimo vselej enake konvergentne rešitve [4]. Nadzorni parameter, ki ga v procesu priraščanja spreminjamo, je pri primarni resonanci in pri subharmonični resonanci tretjega reda fazni kot Φ_1. Pri superharmonični resonanci tretjega reda je kontrolni parameter fazni kot Φ_2, medtem ko bi pri superharmonični resonanci petega reda za nadzorni parameter izbrali fazni kot Φ_3 itd. Velikost prirastkov nadzornih parametrov in območja, v katerih nadzorne parametre spreminjamo, so za posamezno vrsto resonance razvidna iz preglednice 1. Frekvenca nihanja je pri postopku fazne povečave neznana in jo izračunamo obenem z velikostjo. Grafični potek izračunanih resonanc Duffingovega oscilatorja prikazuje slika 1.

Table 1 shows the results of computed primary resonance, superharmonic resonance of the third order, as well as subharmonic resonance of the third order of the Duffing oscillator by using phase incrementation process. The results are checked by computations in which a different, but otherwise sufficiently great number of harmonics is included. Although the number of iterations at the different number of included harmonics differs in general, the same convergent solutions are obtained in all cases [4]. The control parameter, that changes inside the augmentation process, is the phase angle Φ_1 in the case of primary and subharmonic resonance of the third order. The control parameter at superharmonic resonance of the third order is the phase angle Φ_2, whereas at the superharmonic resonance of the fifth order it would be chosen as Φ_3, etc. The ranges of control parameters and corresponding increments can be seen from table 1. Oscillation frequency is unknown in the phase incrementation process and is computed together with magnitude. Plots of computed resonances of the Duffing oscillator are shown in figure 1.

Iz preglednice 1 in s slike 1 izhaja, da se primarna resonanca in subharmonična resonanca tretjega reda pojavljata v okolici frekvence $\omega = 1\,s^{-1}$, superharmonična resonanca tretjega reda pa v okolici frekvence $\omega = \frac{1}{3}\,s^{-1}$, kar se ujema z rezultati, ki jih dobimo z uporabo perturbacijskih metod [1].

From table 1, as well as from figure 1, it follows that primary resonance and subharmonic resonance of the third order appear in the vicinity of frequency $\omega = 1\,s^{-1}$ whereas the superharmonic resonance of the third order appears around frequency $\omega = \frac{1}{3}\,s^{-1}$. This statement agrees with the well known results from the perturbation theory [1].

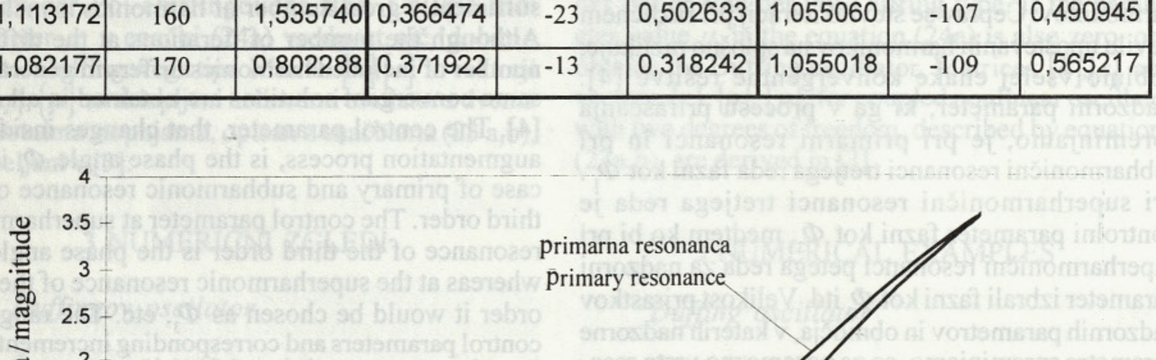
V obravnavanem primeru sta superharmonična in subharmonična resonanca slabo izraženi v primerjavi s primarno resonanco.

In the previously discussed example, superharmonic and subharmonic resonance are more weakly expressed than primary resonance.

Preglednica 1: *Primarna, superharmonična in subharmonična resonanca 3. reda Duffingovega oscilatorja* $(\lambda = -2;\ \varepsilon_1 = 0{,}01;\ \upsilon_1 = 1;\ \mu_1 = 0;\ \alpha_{11} = 0{,}1)$

Table 1: *Primary, superharmonic and subharmonic resonance of 3rd order of Duffing oscillator* $(\lambda = -2;\ \varepsilon_1 = 0.01;\ \upsilon_1 = 1;\ \mu_1 = 0;\ \alpha_{11} = 0.1)$

ω	Φ_1	A_1	ω	Φ_2	A_2	ω	Φ_1	A_1
0,974270	10	0,891113	0,327233	-173	0,139154	1,069688	-77	0,422587
1,065723	20	1,603873	0,348554	-163	0,346251	1,067520	-79	0,388716
1,148446	30	2,174205	0,354790	-153	0,518287	1,065792	-81	0,363690
1,222671	40	2,623257	0,358247	-143	0,653723	1,064364	-83	0,345343
1,285517	50	2,971224	0,360552	-133	0,757667	1,063148	-85	0,332298
1,335470	60	3,231459	0,362181	-123	0,835365	1,062083	-87	0,323683
1,371798	70	3,412075	0,363330	-113	0,890787	1,061131	-89	0,318967
1,394145	80	3,517736	0,364104	-103	0,926606	1,060261	-91	0,317872
1,402379	90	3,550834	0,364571	-93	0,944414	1,059453	-93	0,320336
1,396517	100	3,511986	0,364781	-83	0,944889	1,058693	-95	0,326500
1,376743	110	3,400237	0,364788	-73	0,927871	1,057968	-97	0,336732
1,343462	120	3,212813	0,364655	-63	0,892325	1,057274	-99	0,351681
1,297474	130	2,944497	0,364480	-53	0,836182	1,056612	-101	0,372388
1,240408	140	2,586382	0,364431	-43	0,756063	1,055993	-103	0,400484
1,175895	150	2,123941	0,364836	-33	0,646990	1,055449	-105	0,438568
1,113172	160	1,535740	0,366474	-23	0,502633	1,055060	-107	0,490945
1,082177	170	0,802288	0,371922	-13	0,318242	1,055018	-109	0,565217



Sl. 1. *Primarna, superharmonična in subharmonična resonanca Duffingovega oscilatorja*

Fig.1. *Primary, superharmonic and subharmonic resonance of Duffing oscillator*

Van der Polov oscilator

Vsiljena nihanja pri Van der Polovem oscilatorju dobimo z enačbo (24a) in vrednostmi parametrov $\mu_1 = \upsilon_1 = 1$ in $\alpha_{11} = 0$. Avtonomni Van der Polov oscilator opisuje enačba (24b), če v njej postavimo vrednosti parametrov $\mu_2 = 1$ in $\upsilon_2 = 1$. Pri avtonomnem oscilatorju, je frekvenca ω poleg vseh drugih harmonskih koeficientov dodatna neznanka. Skupno imamo $2K+1$ neznank $\omega, a_{i,1}, \ldots, a_{i,K}; b_{i,1}, \ldots, b_{i,K}$, vendar imamo na voljo samo $2K$ enačb. Zaradi tega vzamemo, da je eden izmed harmonskih koeficientov, na primer koeficient $b_{i,1}$, enak nič in zamenjamo $K+1$ stolpec sistemske matrike $[k]$ z vektorjem $\{q\}$, ki se pojavlja v izrazu $\{q\}\Delta\omega$ na desni strani enačbe (14). S to metodo je izračunana Van der Polova enačba:

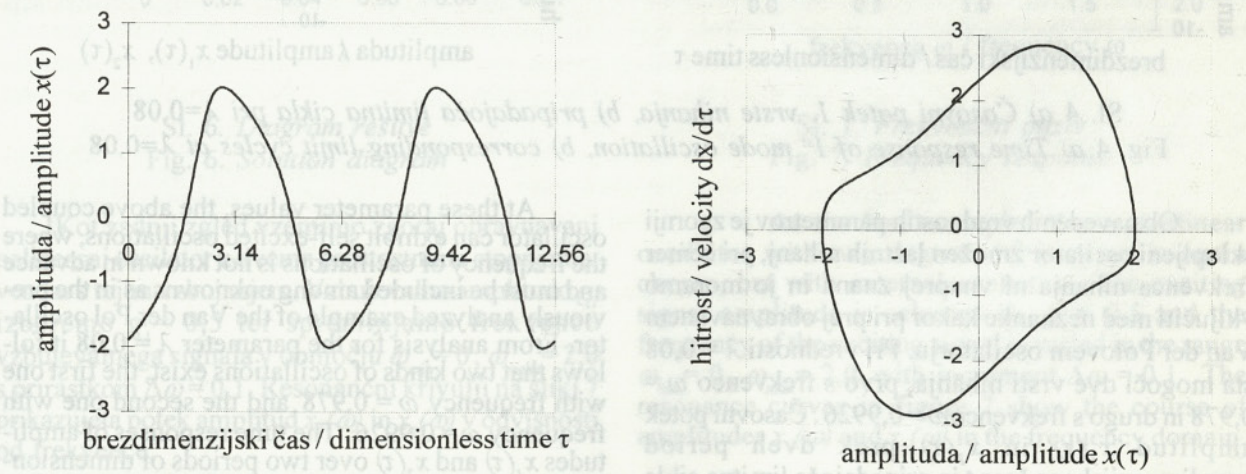
Van der Pol oscillator

Forced vibrations of the Van der Pol oscillator are obtained by equation (24a) and parameter values $\mu_1 = \upsilon_1 = 1$ and $\alpha_{11} = 0$. The autonomous Van der Pol oscillator is described by equation (24b), if parameter values μ_2 and υ_2 in this equation are set equal to 1. The frequency ω of the autonomous oscillator is an additional unknown variable to all other unknown harmonic coefficients. Therefore we have $2K+1$ unknowns $\omega, a_{i,1}, \ldots, a_{i,K}; b_{i,1}, \ldots, b_{i,K}$, but only $2K$ equations. Because of this, one of the harmonic coefficients, for example $b_{i,1}$ is fixed at zero, whereas the $K+1^{st}$ column in the system matrix $[k]$ is interchanged with vector $\{q\}$ which appears in the expression $\{q\}\Delta\omega$ on the right hand side of equation (14). By using this method, Van der Pol equation:

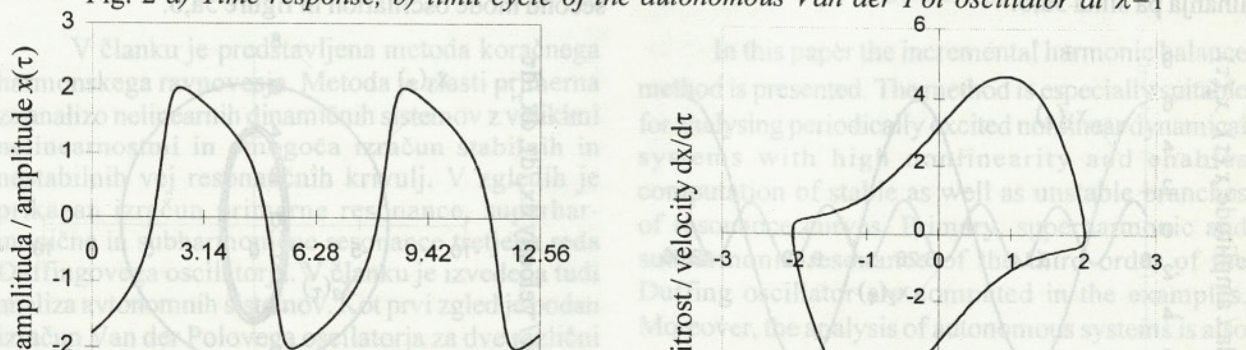
$$\omega^2 \ddot{x} - \lambda\left(1 - x^2\right)\omega \dot{x} + x = 0 \qquad (28),$$

z vrednostima parametrov $\lambda = 1$ in $\lambda = 2$. Izračunana frekvenca pri $\lambda = 1$ znaša $\omega = 0{,}94296$ in $\omega = 0{,}82350$ pri $\lambda = 2$. Na slikah 2 a,b in 3 a,b sta prikazana časovna poteka lastnih oscilacij prek dveh period brezdimenzijskega časa t in pripadajoča limitna cikla za stopnji nelinearnosti $\lambda = 1$ in $\lambda = 2$.

is computed by parameter values $\lambda = 1$ and $\lambda = 2$. The computed frequencies at $\lambda = 1$ are $\omega = 0.94296$ and $\omega = 0.82350$ at $\lambda = 2$. Figures 2 a,b and 3 a,b show the time responses of self excited oscillations over two periods of dimensionless time τ and the corresponding limit cycles for degree of nonlinearity $\lambda = 1$ and $\lambda = 2$ respectively.



Sl. 2 *a) Časovni odziv, b) limitni cikel lastnih nihanj Van der Polovega oscilatorja pri λ=1*
Fig. 2 *a) Time response, b) limit cycle of the autonomous Van der Pol oscillator at λ=1*



Sl. 3. *a) Časovni odziv, b) limitni cikel lastnih nihanj Van der Polovega oscilatorja pri λ=2*
Fig. 3. *a) Time response, b) limit cycle of autonomous Van der Pol oscillator at λ=2*

Sklopljeni nelinearni oscilator z dvema prostostnima stopnjama

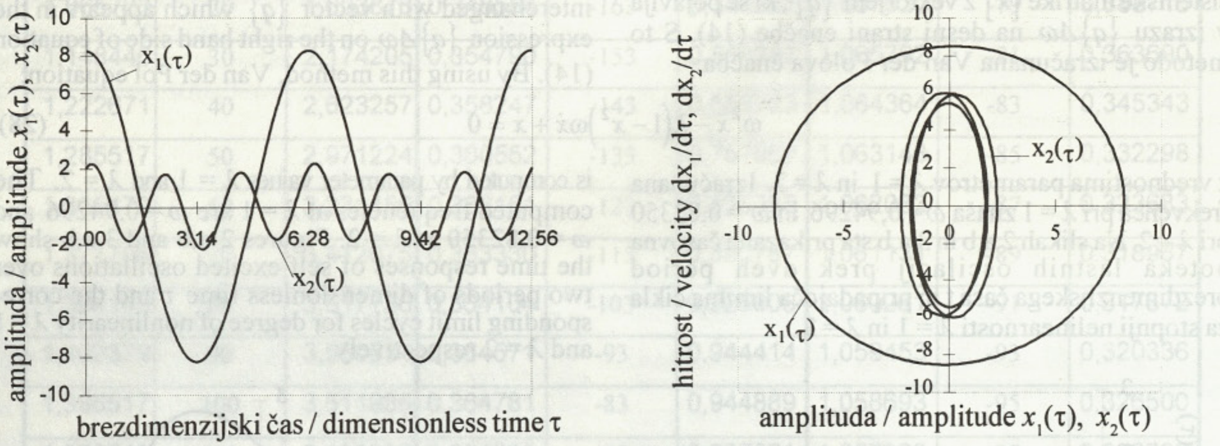
Coupled nonlinear oscillator with two degrees of freedom

Najprej vzemimo avtonomni sklopljeni nelinearni oscilator z dvema prostostnima stopnjama, ki ga lahko popolnoma opišemo z enačbama (23a,b). V tem primeru ni zunanjega vzbujanja, zaradi česar je g_1 v enačbi (23a) enak nič, g_1=0. Preostale vrednosti parametrov izberimo takole:

First we have chosen the autonomous coupled nonlinear oscillator with two degrees of freedom, that can be completelly described by means of equations (23a,b). In this case external excitation is not present, which implies g_1=0 in the equation (23a). Other values of parameters are chosen as follows:

$$\lambda = 0{,}08 \quad \varepsilon_1 = 0{,}2 \quad \varepsilon_2 = -1 \quad \mu_1 = \mu_2 = 0 \quad \upsilon_1 = 1 \quad \upsilon_2 = 9$$

$$\alpha_{11} = 0 \quad \alpha_{12} = -0{,}008 \quad \alpha_{13} = 0 \quad \alpha_{14} = 0 \quad \alpha_{21} = -0{,}0064 \quad \alpha_{22} = \alpha_{23} = \alpha_{24} = 0$$

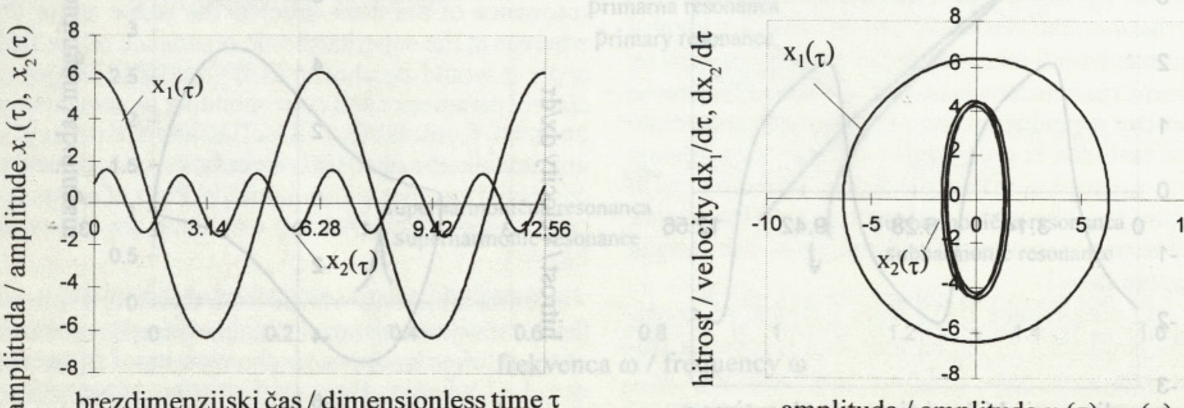


Sl. 4 *a) Časovni potek 1. vrste nihanja, b) pripadajoča limitna cikla pri* λ=0,08

Fig. 4 *a) Time response of 1st mode oscillation, b) corresponding limit cycles at* λ=0,08

Ob navedenih vrednostih parametrov je zgornji sklopljeni oscilator zmožen lastnih nihanj, pri čemer frekvenca nihanja ni vnaprej znana in jo moramo vključiti med neznanke kakor pri prej obravnavanem Van der Polovem oscilatorju. Pri vrednosti $\lambda = 0{,}08$ sta mogoči dve vrsti nihanja, prvo s frekvenco $\omega = 0{,}978$ in drugo s frekvenco $\omega = 0{,}9926$. Časovni potek amplitud $x_1(\tau)$ in $x_2(\tau)$ prek dveh period brezdimenzijskega časa t in pripadajoča limitna cikla pri prvi vrsti nihanja prikazuje slika 4a,b, pri drugi vrsti nihanja pa slika 5a,b.

At these parameter values, the above coupled oscillator can exhibit self-excited oscillations, where the frequency of oscillations is not known in advance and must be included among unknowns as in the previously analyzed example of the Van der Pol oscillator. From analysis for the parameter $\lambda = 0.08$ it follows that two kinds of oscillations exist, the first one with frequency $\omega = 0{,}978$ and the second one with frequency $\omega = 0.9926$. The time responses of amplitudes $x_1(\tau)$ and $x_2(\tau)$ over two periods of dimensionless time τ and the corresponding limit cycles at first mode oscillation are shown in figure 4a,b and for second mode oscillation in figure 5a,b.

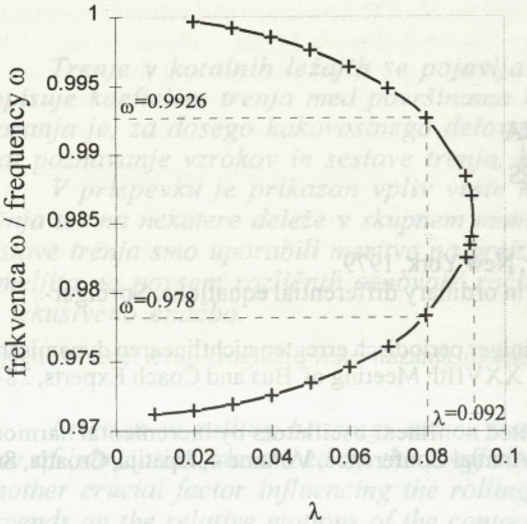


Slika 5 *a) Časovni potek 2. vrste nihanja, b) pripadajoča limitna cikla pri* λ=0,08

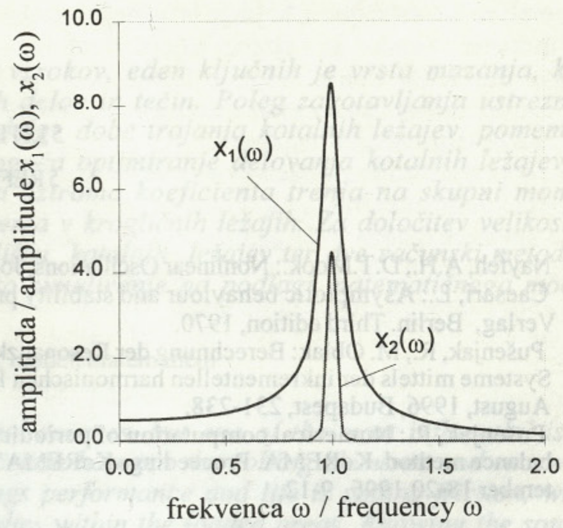
Fig. 5 *a) Time response of 2nd mode oscillation, b) corresponding limit cycles at* λ=0.08

S priraščanjem prostega parametra λ zgradimo celotni diagram rešitve, ki je prikazan na sliki 6. V diagramu sta označeni obe vrsti lastnih periodičnih nihanj ob vrednosti parametra $\lambda = 0{,}08$, pa tudi vrednost parametra $\lambda = 0{,}092$, pri kateri dobimo samo eno vrsto nihanja. Iz diagrama rešitve izhaja, da pri vrednostih parametra $\lambda > 0{,}092$ periodična nihanja ne obstajajo več.

By using augmentation of parameter λ, the complete solution diagram is constructed and plotted on figure 6. In the solution diagram, both kinds of self-excited periodic oscillations for value of parameter $\lambda = 0.08$ are indicated as well as the value of parameter $\lambda = 0.092$, where only one mode of oscillation is obtained. From the solution diagram it follows that at values $\lambda > 0.092$ periodic oscillations do not exist anymore.



Sl. 6. *Diagram rešitve*
Fig. 6. *Solution diagram*



Sl. 7. *Frekvenčni odziv*
Fig. 7. *Frequency response*

Kot zadnji zgled vzemimo zgoraj obravnavani nelinearni oscilator z dvema prostostnima stopnjama, vendar s tujim vzbujanjem. Vzbujevalno amplitudo g_1 izberemo $g_1 = 0{,}5$ ter spreminjamo frekvenco vzbujevalnega signala v območju $\omega_{min} = 0$, $\omega_{maks} = 2{,}0$ s prirastkom $\Delta\omega = 0{,}1$. Resonančni krivulji na sliki 7 prikazujeta potek amplitud $x_1(\omega)$ in $x_2(\omega)$ v odvisnosti od frekvence.

As the last example, above nonlinear oscillator with two degrees of freedom is again chosen, but with external excitation. The exciting signal amplitude is selected as $g_1 = 0.5$ and the frequency of the exciting signal is varied in the range $\omega_{min} = 0$, $\omega_{max} = 2.0$ with increment $\Delta\omega = 0.1$. The resonance curves in figure 7 show the course of amplitudes $x_1(\omega)$ and $x_2(\omega)$ in the frequency domain.

4 SKLEP

V članku je predstavljena metoda koračnega harmonskega ravnovesja. Metoda je zlasti primerna za analizo nelinearnih dinamičnih sistemov z velikimi nelinearnostmi in omogoča izračun stabilnih in nestabilnih vej resonančnih krivulj. V zgledih je prikazan izračun primarne resonance, superharmonične in subharmonične resonance tretjega reda Duffingovega oscilatorja. V članku je izvedena tudi analiza avtonomnih sistemov. Kot prvi zgled je podan izračun Van der Polovega oscilatorja za dve različni vrednosti prostega parametra, ki pomeni merilo stopnje nelinearnosti sistema. V drugem zgledu so analizirana samostojna in tuje vzbujana nihanja sklopljenega nelinearnega oscilatorja z dvema prostostnima stopnjama.

4 CONCLUSION

In this paper the incremental harmonic balance method is presented. The method is especially suitable for analysing periodically excited nonlinear dynamical systems with high nonlinearity and enables computation of stable as well as unstable branches of resonance curves. Primary, superharmonic and subharmonic resonance of the third order of the Duffing oscillator are computed in the examples. Moreover, the analysis of autonomous systems is also carried out. In the first example, the Van der Pol oscillator with two different levels of nonlinearity is computed. In the second example, self excited as well as forced oscillations of the coupled nonlinear oscillator with two degrees of freedom are analyzed.

Zahvala

Delo je finančno podprlo Ministrstvo za znanost in tehnologijo Slovenije v okviru raziskovalnega projekta: Optimalno projektiranje oblike linijskih konstrukcij z nelinearnim odzivom, štev.: J2-7631-0795/96-98, katerega nosilec je bil prof. dr. M. Oblak.

Aknowledgement

The present work is sponsored by the Ministry of Science and Technology of Slovenia in the framework of the research project: Shape optimal design of plane structures with nonlinear response, No.: J2-7631-0795/96-98 under the leadership of Prof. Dr. M. Oblak.

5 LITERATURA

5 REFERENCES

[1] Nayfeh, A.H., D.T.Mook.: Nonlinear Oscillations, John Wiley, New York, 1979.

[2] Caesari, L.: Asymptotic behaviour and stability problems in ordinary differential equations. Springer-Verlag, Berlin. Third edition, 1970.

[3] Pušenjak, R., M. Oblak: Berechnung der Resonanzkurven einiger periodisch erregten nichtlinearen dynamischen Systeme mittels der inkrementellen harmonischen Balance. XXVIIth Meeting of Bus and Coach Experts, 28-30 August, 1996, Budapest, 231-238.

[4] Pušenjak, R.: Numerical computation of periodically excited nonlinear oscillators by incremental harmonic balance method. KoREMA. Proceedings KoREMA'96, 41st Annual Conference, Volume 4, Opatija, Croatia, September 18-20,1996, 9-12.

Avtorjev naslov: prof. dr. Rudi Pušenjak, dipl. inž.
Univerza v Mariboru
Fakulteta za strojništvo
Smetanova 17
2000 Maribor

Author's Address: Prof. Dr. Rudi Pušenjak, Dipl. Ing.
University of Maribor
Faculty of Mechanical Engineering
Smetanova 17
2000 Maribor, Slovenia

Prejeto:
Received: 3.3.1997

Sprejeto:
Accepted: 30.6.1997