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Effect of asymmetry on the steady thermal convection in a vertical torus filled with a porous medium

Postelnicu Adrian[1], Scurtu Nicoleta[2]

[1]*Department of Thermal Engineering and Fluid Mechanics, Transilvania University, Bdul Eroilor, No 29, Brasov, Romania* e-mail: adip@unitbv.ro*
[2]*Numerical Mathematics and Scientific Computing, Weierstrass-Institute for Applied Analysis and Stochastic (WIAS), Mohrenstrasse 39, 10117 Berlin, Germany*

Abstract

The paper deals with a study on the thermal convection in a fluid saturated porous medium confined in a vertical oriented toroidal loop heated from below and cooled from above, subjected to a vertically uniform temperature gradient. In fact, Sano, O., Journal of the Physical Society of Japan (1987 and 1988) studied such a configuration for clear fluids. We take the steady heat conduction state under a constant vertical temperature gradient ($-k$) as a fundamental solution. A cylindrical system of co-ordinates (r, ϕ, s) is considered for the loop of $2\pi R$ length and cross sectional diameter $2a$, where the ratio a/R is assumed to be very small. We expand all quantities in terms of double Fourier series in ϕ and θ, where $\theta = s/R$. First-order perturbed fields from steady heat conduction state are examined and various plots are given: isotherms in ring and meridian plane, stream lines and velocities. Finally, an attempt is made in order to identify other fundamental solutions: S-type (cellular thermal convection) and A-types (coaxial flow, bidirectional flow and antisymmetric cellular flow).

Introduction

Thermal convection in a vertically oriented torus has various applications in engineering and nature, like cooling systems in nuclear engineering, solar heaters geothermal engineering, etc. Several papers considered this configuration, in Newtonian fluids, see for instance [1] and [2].

On the other hand, natural convection in porous media is of interest in many applications and recent books by Nield and Bejan [3] and Ingham and Pop [4-5] present a comprehensive account of the available information in the field. For example, natural convection in a horizontal porous annulus is well documented in the literature, a whole chapter in the reference [5] being dedicated to this theme. However, studies on thermal convection in toroidal configurations filled with fluid-saturated porous media seem to be scarce in the open literature. One example is the work [6] where it is investigated the natural convection and its stability in a toroidal thermosyphon filled with a porous medium. The onset of the thermal convection in that configuration was studied using a one-dimensional model.

We remark at this point that the stability of flows in porous media differs considerably from that of Newtonian fluids, due to the very specific changes in the hydraulic and thermal properties.

The objective of the present paper is to study the steady thermal convection in a vertical torus filled with a fluid-saturated porous medium. The fluid in the porous medium is considered incompressible and obeying the Boussinesq law. The loop is heated from below and cooled from above and it is subjected to a vertically uniform temperature gradient

Mathematical formulation

We consider a thermal convection in a vertical torus heated from below and cooled from above. We denote the cross sectional diameter of the torus by $2a$ and its loop length by $2\pi R$, where the ratio a/R is assumed to be very small. The x and z axis are taken in the plane of the generator of the torus, with the z axis in the opposite direction of the gravity, see Fig. 1. The reference system (r, ϕ, s) is also introduced, where s is the distance measured counterclockwise from the bottom, along the generator, while r and ϕ are polar co-ordinates of the cross section. The direction $\phi = 0$ is chosen so that it always coincides with the direction from the generator to the outer edge of the torus in the ring xz plane.

Nomenclature

a	half cross-sectional diameter of the torus		β	thermal expansion coefficient
k	vertical temperature gradient		ϕ	polar co-ordinate
K	permeability of the porous medium		μ	dynamic viscosity of the fluid
g	acceleration of gravity		ν	kinematic viscosity of the fluid
p	pressure		ρ	density of the fluid
r	polar co-ordinate		θ	$= s/R$
Ra	Rayleigh number			
(u, v, w)	components of the velocity			**Subscripts**
s	distance measured counterclockwise from the bottom		0	reference conditions
T	temperature			
z	vertical Cartesian co-ordinate			**Superscripts**
			B	base solution for symmetric modes
	Greek symbols		'	perturbed quantities
α	thermal diffusivity		*	non-dimensional quantities

We analyse the fluid motion through the porous medium on the basis of the following assumptions:

- the flow is steady, laminar and incompressible;
- the liquid and solid matrix are in thermal equilibrium;
- properties of the fluid and of porous matrix are constant except for the density of the fluid which depends on the temperature according to the Boussinesq approximation.

The governing equations for the velocity $\mathbf{v}$, temperature T and pressure p are

$$\nabla \cdot \mathbf{v} = 0 \tag{1}$$

$$\mathbf{v} = -\frac{K}{\mu}\left(\nabla p + \rho_0 \beta g T \mathbf{e}_z\right) \tag{2}$$

$$\mathbf{v} \cdot \nabla T = \alpha \nabla^2 T \tag{3}$$

where ρ_0 is the density of the fluid at the reference temperature T_0, g is the gravitational acceleration, $\mathbf{e}_z$ is an upward unit vector, μ, β, α and K are the dynamic viscosity of the fluid, thermal expansion coefficient, thermal diffusivity and permeability of the porous medium, respectively. We take the steady state as heat conduction under a constant vertical temperature gradient ($-k$), which is in fact a fundamental solution

$$T = T_0 - kz \tag{4}$$

$$p = p_0 - \rho_0 g z - \frac{1}{2}\rho_0 \beta k g z^2 \tag{5}$$

The perturbed fields $\mathbf{v}'$, T' and p' satisfy the following equations, which are correct to the first order

$$\nabla \cdot \mathbf{v}' = 0 \tag{6}$$

$$\mathbf{v}' = -\frac{K}{\mu}\left(\nabla p' + \rho_0 \beta g T' \mathbf{e}_z\right) \tag{7}$$

$$v_z' k = \alpha \nabla^2 T' \tag{8}$$

The boundary conditions are

$$\mathbf{v}' = T' = 0 \text{, at } r = a \tag{9}$$

Introducing the non-dimensional quantities

$$\mathbf{x}^* = \frac{1}{a}\mathbf{x}, \quad \mathbf{v}^* = \frac{a}{\alpha}\mathbf{v}', \quad p^* = \frac{p' K}{\alpha \mu}, \quad T^* = \frac{T'}{ka} \tag{10}$$

equations (1-3) become

$$\nabla^* \cdot \mathbf{v}^* = 0 \tag{11}$$

$$\mathbf{v}^* = -\nabla^* p^* + Ra T^* \mathbf{e}_z \tag{12}$$

$$v_z^* = -\Delta^* T^* \tag{13}$$

where (3-4) have been used. Further, $Ra = k\beta g K a^2/(\nu\alpha)$ is the Rayleigh number. The boundary conditions, in dimensionless form, read

$$\mathbf{v}^* = T^* = 0 \text{, at } r^* = 1 \tag{14}$$

Expressed in terms of the (r, ϕ, s) coordinate system, where the velocity components are u^*, v^* and w^* and dropping the asterisks for convenience, the governing equations are

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial w}{\partial \phi} + \frac{1}{R}\frac{\partial w}{\partial \theta} = 0 \tag{15}$$

$$u = -\frac{\partial p}{\partial r} - RaT\cos\theta\cos\phi \tag{16}$$

$$v = -\frac{1}{r}\frac{\partial p}{\partial \phi} + RaT\cos\theta\sin\phi \tag{17}$$

$$w = RaT\sin\theta \tag{18}$$

$$(u\cos\phi - v\sin\phi)\cos\theta - w\left(1+\frac{r}{R}\cos\phi\right)\sin\theta = \Delta T \tag{19}$$

where $\Delta = -\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{1}{R^2}\frac{\partial^2}{\partial \theta^2}$ is the Laplacean and $\theta = s/r$. Collecting the zeroth order of a/R (<<1), equations (15)-(19) become

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial w}{\partial \phi} + \frac{1}{R}\frac{\partial w}{\partial \theta} = 0 \tag{20}$$

$$u = -\frac{\partial p}{\partial r} - RaT\cos\theta\cos\phi \tag{21}$$

$$v = -\frac{1}{r}\frac{\partial p}{\partial \phi} + RaT\cos\theta\sin\varphi \tag{22}$$

$$w = RaT\sin\theta \tag{23}$$

$$(u\cos\phi - v\sin\phi)\cos\theta - w\sin\theta = \Delta T \tag{24}$$

where $\Delta = -\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + O(R^{-2})$, so that these expressions are the same as those given in terms of a straight circular cylindrical coordinate systems, except that directions of v_z and e_z change at different positions along the generator. We confine our attention to the steady thermal convection, which corresponds to neutrally stable states at some particular critical Rayleigh numbers.

Some fundamental solutions for symmetric modes

In this paper we are mainly interested in the analysis of symmetric modes which are caused by experimentally uncontrollable small disturbances under symmetric boundary conditions. The procedure is similar with that used by Sano in [1-2], by expanding all relevant quantities in terms of double Fourier series in ϕ and θ

$$(T, p, u) = \sum_{m,n=0}^{\infty}\{T_{m,n}(r), p_{m,n}(r), u_{m,n}(r)\} \cdot \cos m\phi\cos n\theta, \tag{25}$$

$$v = \sum_{m=0,n=1}^{\infty} v_{m,n}(r)\sin m\phi\cos n\theta,$$

$$w = \sum_{m=0,n=1}^{\infty} w_{m,n}(r)\cos m\phi\sin n\theta$$

Results for symmetric modes

Due to the fact that temperature distributions with m = 0 and n = 0 do not lead to physically realizable symmetric convection in a vertically oriented torus, we focus on the simplest situation, represented by $T_{1,1}$, $w_{1,2}$, $p_{1,2}$, $u_{1,2}$ and $v_{1,2}$ series. By truncating these series at the lowest order, we have

$$(u_{1,2})' + \frac{1}{r}(u_{1,2} + v_{1,2}) + \frac{2}{R}w_{1,2} = 0 \tag{26}$$

$$u_{1,2} + (p_{1,2})' = 0 \tag{27}$$

$$v_{1,2} = \frac{1}{r}p_{1,2} \tag{28}$$

$$-w_{1,2} = 2\left[(T_{1,1})'' + \frac{1}{r}(T_{1,1})' - \frac{1}{r^2}T_{1,1}\right] \tag{29}$$

$$w_{1,2} = \frac{Ra}{2}T_{1,1} \tag{30}$$

$$T_{1,1} = u_{1,2} = v_{1,2} = w_{1,2} = p_{1,2} = 0, \text{ at } r = 1 \tag{31}$$

The solution of the set of equations (26-30) is obtained, after some algebra, in the form

$$T_{1,1} = AJ_1(k_n r) + BN_1(k_n r) \tag{32a}$$

$$u_{1,2} = -c_1 + \frac{c_2}{r^2} + A\{[J_0(k_n r) - J_2(k_n r)] + B[N_0(k_n r) - N_2(k_n r)]\}k_n \tag{32b}$$

$$v_{1,2} = c_1 + \frac{c_2}{r^2} - \frac{4}{rR}[AJ_1(k_n r) + BN_1(k_n r)] \tag{32c}$$

$$w_{1,2} = 2k_n^2[AJ_1(k_n r) + BN_1(k_n r)] \tag{32d}$$

$$p_{1,2} = c_1 r + \frac{c_2}{r^2} - \frac{4}{R}[AJ_1(k_n r) + BN_1(k_n r)] \tag{32b}$$

where J_n are Bessel functions of the first kind of nth order, N_n are Bessel functions of the second kind and nth order, c_1, c_2, A and B are constants and $k_n = Ra^{1/2}/4$.

Imposing the boundary conditions (31), the solutions (32) can be expressed in the form

$$u = u^B \cos\phi \cos 2\theta\,,\ v = v^B \sin\phi \cos 2\theta\,,\quad (33)$$
$$w = w^B \cos\phi \sin 2\theta\,,\ p = p^B \cos\phi \cos 2\theta\,,$$
$$T = T^B \cos\phi \cos\theta$$

where

$$T^B = T_{1,1} = AJ_1(k_{1l} r) \quad (34a)$$

$$u^B = u_{1,2} = \frac{A}{R}\left[-2\left(1+\frac{1}{r^2}\right)J_0(k_{1l} r) + 2k_{1l}\left(J_0(k_{1l} r) - J_2(k_{1l} r)\right)\right] \quad (34b)$$

$$v^B = v_{1,2} = \frac{A}{R}\left[2\left(1-\frac{1}{r^2}\right)J_0(k_{1l} r) - \frac{4}{r}J_1(k_{1l} r)\right] \quad (34c)$$

$$w^B = w_{1,2} = 2k_{1l}{}^2 AJ_1(k_{1l} r) \quad (34d)$$

$$p^B = p_{1,2} = \frac{A}{R}\left[2\left(r-\frac{1}{r}\right)J_0(k_{1l} r) - 4J_1(k_{1l} r)\right] \quad (34e)$$

The solution is valid for $Ra = 4k_{1l}^2$, where k_{1l} (l = 1, 2, 3, …) are the zeroes of J_1, whilst A is a undetermined constant. The first zero of J_1 is k_{1l} = 3.831706, so that the critical Rayleigh number is Ra_1 = 58.72788.

Fig. 2 shows the isotherms in the ring plane (*xz*-plane), $T = T^B(r)\cos\theta$ = const. Next, in Fig. 3 there are plotted the temperature distributions in the meridian plane ($\theta = 0$, or $\theta = \pi$). The streamlines in the ring plane (*xz*-plane) are represented in Fig. 4. We notice that in the *xz*-plane $\mathbf{v} = (u, 0, w)$, so the flow is assumed as two-dimensional. Such a plot for clear fluids, based on the same assumption, can be found in Sano [1]. We can use here the same argument as there: such as a stream function gives good qualitative patterns for the flow in the loop with small a/R, when the w component exceeds the others in the most part of the porous medium. Finally, Fig. 5 shows the (u, v) velocity fields, in vector representation, in the meridian plane ($\theta = 0$ or $\theta = \pi$).

All the previous results belong to the so-called S-type modes, specifically denoted by S_1^2. Type S has also cellular thermal convection at $Ra = 2k_{1l}^2$, denoted by S_1^1 and is defined by

$$u = 0.5u^B \cos\phi \cos\theta\,,\quad (35)$$
$$v = 0.5v^B \sin\phi \cos\theta\,,\ w = w^B \cos\phi \sin\theta\,,$$
$$p = 0.5p^B \cos\phi \cos\theta\,,\ T = T^B \cos\phi$$

Some fundamental solutions for antisymmetric modes

There are several types of antisymmetric modes, as for clear fluids, see [2]. Let us begin with the A-type modes, characterized by

$$(T, p, u) = \sum_{m,n=0}^{\infty}\{T_{m,n}(r), p_{m,n}(r), u_{m,n}(r)\} \cdot \cos m\phi \sin n\theta\,,\quad (36)$$
$$v = \sum_{m=0,n=1}^{\infty} v_{m,n}(r) \sin m\phi \sin n\theta\,,$$
$$w = \sum_{m=0,n=1}^{\infty} w_{m,n}(r) \cos m\phi \cos n\theta$$

This type include a family of coaxial flow along the loop at $Ra = 2k_{0l}^2$, which is denoted as in [2] by A_0^0. The distribution of the relevant quantities in this case is

$$u = v = 0\,, w = Ak_{0l}^2 J_0(k_{0l} r),\quad (37)$$
$$T = AJ_0(k_{0l} r)\sin\theta$$

where l = 1, 2, …

Another A-type of solution corresponds to a bidirectional flow at $Ra = 2k_{1l}^2$, denoted by A_1^0. In this case,

$$u = v = 0\,, w = w^B \cos\phi\,,\quad (38)$$
$$T = T^B \cos\phi \sin\theta$$

At $Ra = 4k_{1l}^2$, where l = 1, 2, …, we obtain an antisymmetric cellular flow, called A_1^2. Now the distributions are

$$u = u^B \cos\phi \sin 2\theta\,,\quad (39)$$
$$v = v^B \sin\phi \sin 2\theta\,, w = -w^B \cos\phi \cos 2\theta\,,$$
$$p = p^B \cos\phi \sin 2\theta\,,\ T = T^B \cos\phi \sin\theta$$

Other fundamental solutions

By inverting the sine and cosine terms in ϕ for S- and A-mode types, we can obtain other fundamental solutions, see also Sano [2], for clear fluids.

The $\tilde{S}$ -type is obtained from (25) as follows

$$(T, p, u) = \sum_{m,n=0}^{\infty}\{T_{m,n}(r), p_{m,n}(r), u_{m,n}(r)\} \cdot \sin m\phi \cos n\theta\,,\quad (40)$$
$$v = \sum_{m=0,n=1}^{\infty} v_{m,n}(r) \cos m\phi \cos n\theta\,,$$

$$w = \sum_{m=0,n=1}^{\infty} w_{m,n}(r) \sin m\phi \sin n\theta$$

The $\widetilde{A}$ -type is obtained from (25) in the form

$$(T, p, u) = \sum_{m,n=0}^{\infty} \{T_{m,n}(r), p_{m,n}(r), u_{m,n}(r)\} \cdot \sin m\phi \sin n\theta , \quad (41)$$

$$v = \sum_{m=0,n=1}^{\infty} v_{m,n}(r) \cos m\phi \sin n\theta ,$$

$$w = \sum_{m=0,n=1}^{\infty} w_{m,n}(r) \sin m\phi \cos n\theta$$

The fundamental solutions are readily obtained from their counterparts in S- and A-modes, according to the basic rule stated above (inverting the sine and cosine terms in ϕ). For example, the fundamental solution for $\widetilde{A}_1^2$ -type flow is obtained from (39) as

$$u = u^B \sin\phi \sin 2\theta , \quad (42)$$
$$v = v^B \cos\phi \sin 2\theta , \; w = -w^B \sin\phi \cos 2\theta ,$$
$$p = p^B \sin\phi \sin 2\theta , \; T = T^B \sin\phi \sin\theta .$$

Conclusion

An analysis of steady thermal convection in a vertical torus filled with fluid-saturated porous medium was presented in this paper. Many similarities were found between this physical case and that of vertical torus filled with Newtonian fluid [1-2].

Taking the steady state as heat conduction under a constant vertical temperature gradient, all the perturbed quantities have been expanded in double Fourier series but only the lowest order terms have been retained in the analysis. The main reason in doing so was to keep close the procedure to that one used in clear fluids [1-2], in order to facilitate (qualitative) comparisons, and this line of study gave fruitful results.

It is worth to check the influence of higher modes, given by increasing of the truncated terms in the Fourier expansions. A further step is the superposition of the antisymmetric modes, which can be performed similarly as in [2]. The results of these tasks will be presented elsewhere.

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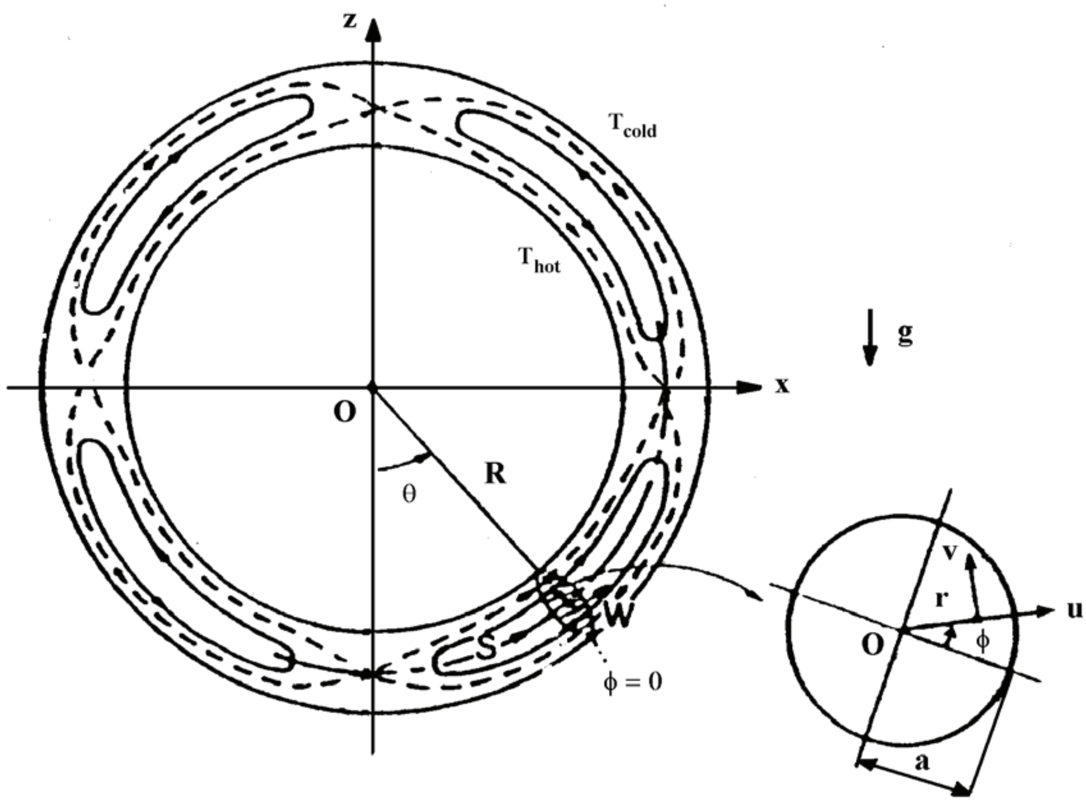

Fig. 1. Sketch of the physical problem.

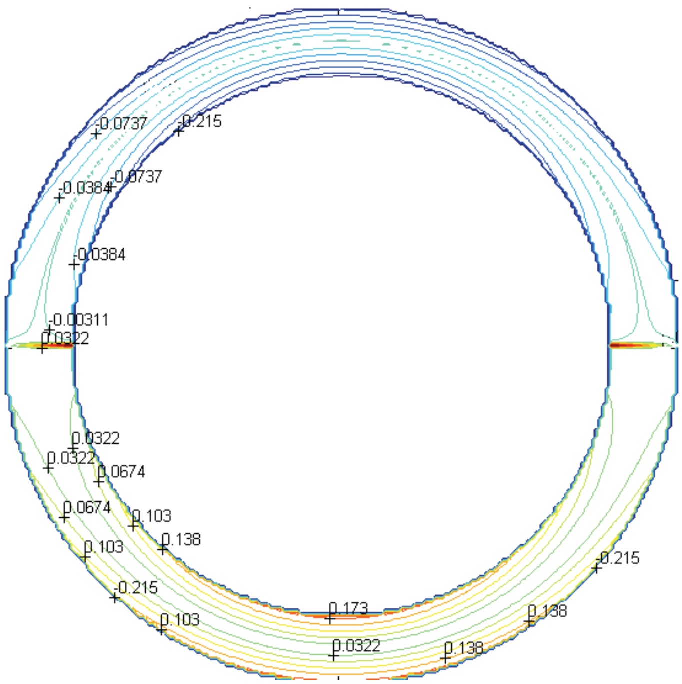

Fig. 2 Isotherms in the ring plane (*xz*-plane).

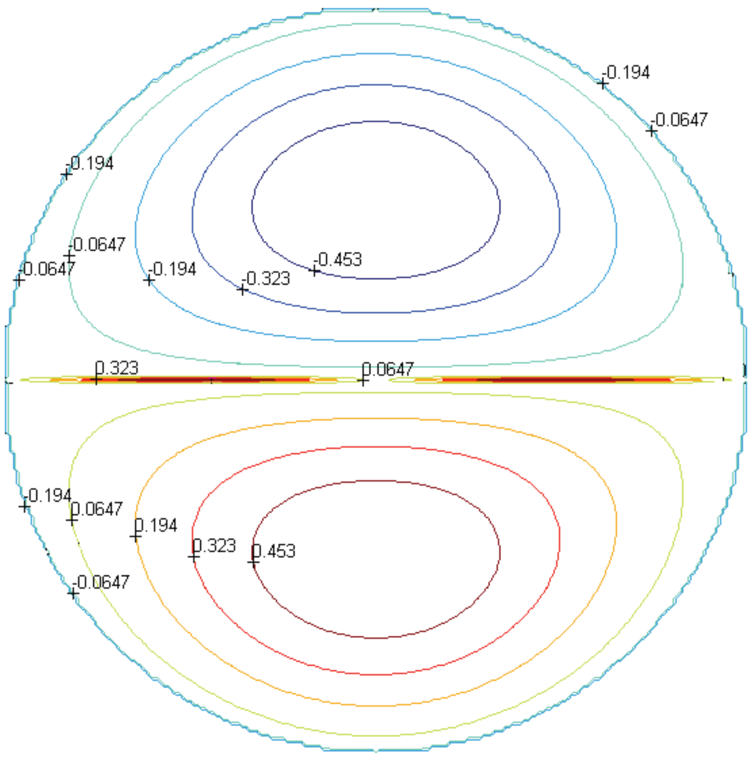

Fig. 3 Isotherms in the meridian plane (θ = 0 or θ = π).

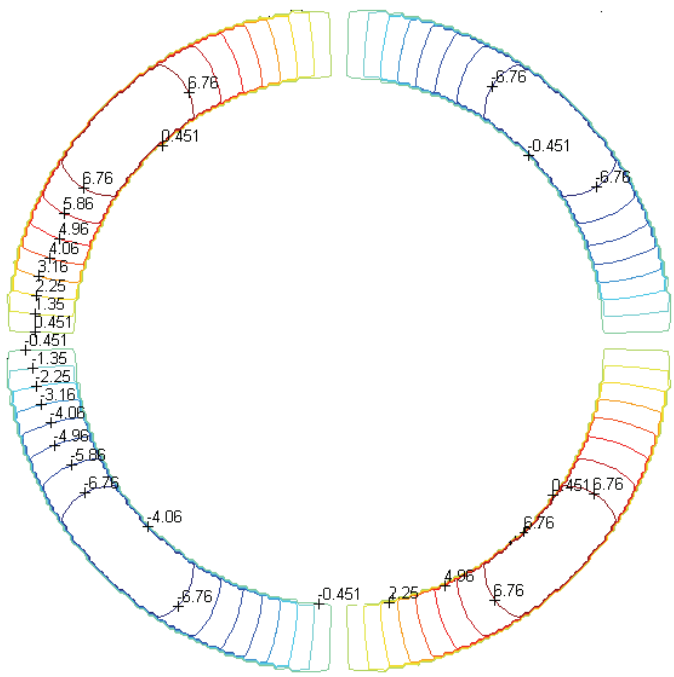

Fig. 4 Streamlines in the ring plane (*xz*-plane).

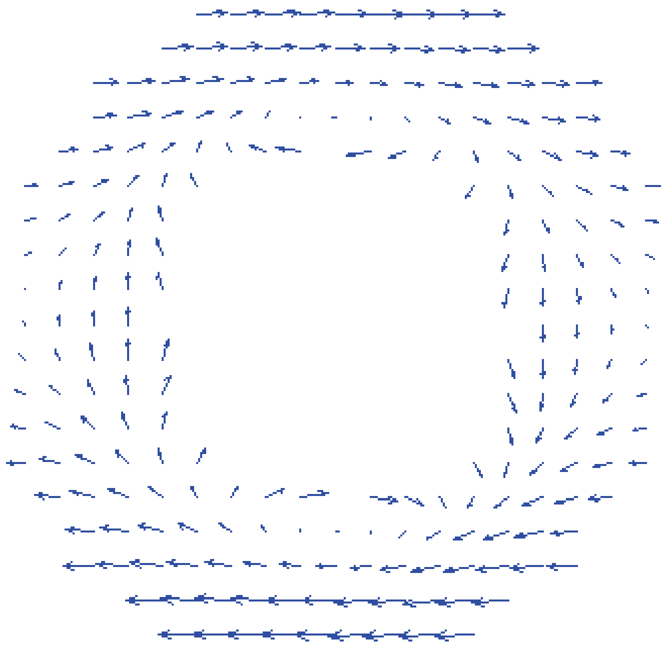

Fig. 5. (u, v) velocity fields in the meridian plane ($\theta = 0$ or $\theta = \pi$).