An Approach to the Optimization of a Thin-walled Z-beam

Nina Anđelić[*] - Vesna Milošević Mitić - Taško Maneski
University of Belgrade, Faculty of Mechanical Engineering, Serbia

One approach to the optimization of a thin-walled open section Z-beam subjected to bending and to the constrained torsion is considered. For given loads, material and geometrical characteristics the problem is reduced to the determination of minimum mass i.e. minimum cross-sectional area of a structural thin-walled beam of a chosen shape. The area of the cross section is assumed to be the objective function. The stress constraints are introduced. A general case when bending moments about two centroidal axes and the bimoment are acting simultaneously is derived, and then some particular loading cases are considered. A method of solving the optimal relation of the parts of the considered cross-section is described. Applying the Lagrange multiplier method, the equations, whose solutions represent the optimal values of the ratios of the parts of the chosen cross-section, are formed. The obtained results are used for numerical calculation.

0 INTRODUCTION

Many modern metal structures (motor and railroad vehicles, naval structures, turbine blades) are manufactured using thin-walled elements (shells, plates, thin-walled beams) which are subjected to complex loads. In most structures it is possible to find the elements in which, depending on loading cases and the way they are introduced, the effect of constrained torsion is present and its consequences are particularly evident in the case of thin-walled profiles.

Investigations of the behaviour of thin-walled members with open cross-sections have been carried out extensively since the early works of S. P. Timoshenko [1], who was among the first to publish a number of books on materials strength, the theory of elasticity and the theory of stability, and who also developed the theory of beams and plates bending. V. Z. Vlasov [2] contributed largely to the theory of thin-walled structures by developing the theory of thin-walled open section beams. Kollbruner and Hajdin [3] and [4] expanded the field of thin-walled structures by a range of their works. Exceptionally valuable among them are two monographs [3] and [4] which with a series of contributions originally describing the authors' area of interest, constitute a unique work in terms of content,. Also, Murray [5] and Rhodes et al. [6] should be mentioned as the authors who introduced the theoretical aspects of the

behaviour of thin-walled structures. Due to their low weight, thin-walled open section beams are widely applied in many structures. Thin-walled beams have a specific behaviour, which is the reason why their optimization is a particular problem.

Analyzing the process of designing various types of structures, it can be observed that the classical procedure of defining dimensions of a structure based on the theory of the strength of materials provides sufficient, however, not the optimum geometric parameters. In general, the optimization is a mathematical process through which a set of conditions, which give the maximum or minimum value of a specified function as a result, are obtained. In the ideal case the perfect solution for the considered design situation is supposed to be obtained but in reality only the best solution although not the perfect one, can be found. Among the authors who developed theoretical fundamentals of the optimization method, Fox [7], Brousse [8], Prager [9] and Rozvany [10] should be given the most prominent place.

Many studies have been conducted on optimization problems, treating the cases where geometric configurations of structures are specified and only the dimensions of structural members and the areas of their cross-sections are determined in order to attain the minimum structural weight or cost [11] and [12]. Tian et al. [13] present a combined theoretical and

[*]Corr. Author's Address: Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Belgrade 35, Serbia, nandjelic@mas.bg.ac.rs

experimental study on the minimum weight and the associated optimal geometric dimensions of an open-channel steel section.

Many authors, including Farkas [14], applied mathematical problems to the conditional extreme of the function with more variables onto the cross-sectional area of the structure and defined optimum cross-section from the aspect of load and consumption of the material. Then, a series of works appear where the problem of optimization of various cross-sections, such as triangular cross-section [15], I-section [16] and [17] or channel-section beams [18] are solved by using the Lagrange multiplier method.

The main purpose of this paper is to present one approach to the optimization of a thin-walled Z-section beam.

1 PROBLEM FORMULATION

The starting points during the formulation of the basic mathematical model are the assumptions of the thin-walled-beam theory, on one hand, and the basic assumptions of the optimum design on the other.

During the process of structure dimensioning, apart from defining the requested dimensions necessary to permit the particular part of the structure to support the applied loads, it is often of significance to determine the optimal values of the dimensions. The Z - cross section as a very often used thin-walled profile in steel structures is in this paper considered as the object of the optimization. The determination of its optimal dimensions is a very important process but not always the simplest one. The aim of the paper is to determine the minimum mass of the whole beam, i.e. the minimum area A of the cross-section of the considered beam for the given loads and material properties.

The cross-section of the considered beam (Fig. 1) with principal centroidal axes X_i ($i = 1, 2$) has the centre and not the axis of symmetry. It is assumed that its flanges have equal widths $b_1 = b_3$, and thicknesses $t_1 = t_3$, and that its web has the width b_2 and thickness t_2. The ratios of thicknesses and widths of flanges and web are treated as non-constant quantities.
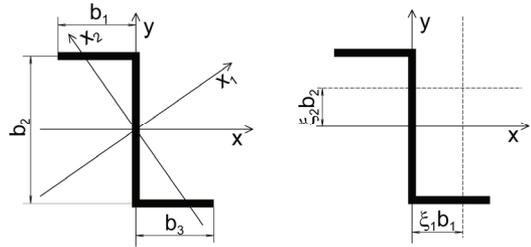
Fig. 1. *Cross-section*

It is also assumed that the loads are applied in two longitudinal planes, parallel to the centroidal axes x and y at the distances $\xi_i b_i$ ($i = 1, 2$) (Fig. 1). If applied in such a way, the loads will cause the bending moments acting in the above mentioned two planes parallel to the longitudinal axis of the beam, and consequently the effects of the constrained torsion will occur in the form of the bimoment, causing the stresses that depend on the boundary conditions [3] and [19].

The aim of the paper is to determine the minimal mass of the beam or, in other words, to find the minimal cross-sectional area

$$A = A_{min} \qquad (1)$$

for the given loads and material and geometrical properties of the considered beam, while satisfying the constraints.

The formulation of the structural design optimization problem plays an important role in the numerical solution process [6]. A particular choice of the objective function and constraints affects the final solution, and the efficiency and robustness of the solution process.

1.1 Objective Function

The process of selecting the best solution from various possible solutions must be based on a prescribed criterion known as the objective function. In the considered problem the cross-sectional area will be treated as an objective function and it is evident from the Fig. 1 that

$$A = \sum b_i t_i, \quad i = 1, 2, 3$$

or (because $b_1 = b_3$)

$$A = A(b_1, b_2) = 2 b_1 t_1 + b_2 t_2. \qquad (2)$$

The thickness-length ratios of the cross-sectional walls are assumed to be non-constant variables (Fig. 1)

$$\frac{t_i}{b_i} \neq const, \quad i = 1, 2, 3 , \tag{3}$$

where t_i and b_i are thickness and widths of flanges and web.

1.2 Constraints

The formulation is restricted to the stress analysis of thin-walled beams with open sections.

Only normal stresses will be taken into account in the consideration that follows and the constraints treated in the paper are the stress constraints. The expressions (4) and (5) for equivalent bending moments [20] taking into account the influence of the bending moments around centroidal axes x and y, denoted as M_x and M_y respectively, will be used

$$\overline{M_x} = \frac{M_x - M_y\left(\dfrac{I_{xy}}{I_y}\right)}{1 - \dfrac{I_{xy}^2}{I_x \cdot I_y}} \tag{4}$$

and

$$\overline{M_y} = \frac{M_y - M_x\left(\dfrac{I_{xy}}{I_x}\right)}{1 - \dfrac{I_{xy}^2}{I_x \cdot I_y}} , \tag{5}$$

where I_x, I_y are the moments of inertia of the cross-sectional area about the centroidal axes x and y, and I_{xy} is the product of inertia.

The normal stresses are caused by the bending moments $\overline{M_x}$ and $\overline{M_y}$ and by the bimoment B that appears in the case of constrained torsion, and they will be denoted as $\overline{\sigma_x}$ and $\overline{\sigma_y}$ and σ_ω respectively [3] and [19].

Bimoment is not a static value, and can not be defined by static equilibrium conditions. When the bending moments act in planes parallel to the longitudinal axis (Fig. 1) at the distances $\xi_i b_i$ (i = 1, 2) the bimoment will appear as a consequence, and it can be expressed as the function of the bending moments and the eccentricities of their planes $\xi_i b_i$ (i = 1, 2) in the following way [3] and [19]

$$B = \xi_1 b_1 \overline{M_x} + \xi_2 b_2 \overline{M_y} . \tag{6}$$

For the allowable stress σ_0 the constraint function can be written as

$$\varphi = \varphi(\sigma) = \overline{\sigma_x}_{\max} + \overline{\sigma_y}_{\max} + \sigma_{\omega\max} \leq \sigma_0 . \tag{7}$$

The maximal normal stresses, are defined in the form [3] and [19]

$$\overline{\sigma_x}_{\max} = \frac{\overline{M_x}}{W_x}, \quad \overline{\sigma_y}_{\max} = \frac{\overline{M_y}}{W_y}, \tag{8}$$

$$\sigma_{\omega\max} = \frac{B}{W_\omega}, \tag{9}$$

where W_x and W_y are the section moduli for the longitidunal axes, and W_ω is the sectorial section modulus for the considered cross-section.

After the introduction of Eqs. (8) and (9) into Eq. (7), the constraint function becomes

$$\varphi = \frac{\overline{M_x}}{W_x} + \frac{\overline{M_y}}{W_y} + \frac{B}{W_\omega} \leq \sigma_0 . \tag{10}$$

The constraint function Eq. (10) is reduced to:

$$\varphi = \varphi(b_1, b_2) =$$

$$30 M_x \frac{1}{t_1 b_1 b_2\left(3 + 2\dfrac{t_2}{t_1}\dfrac{b_2}{b_1}\right)} +$$

$$3 M_y \cdot z \cdot \frac{9 + \dfrac{t_2}{t_1}\dfrac{b_2}{b_1}}{t_1 b_1 b_2\left(3 + 2\dfrac{t_2}{t_1}\dfrac{b_2}{b_1}\right)} +$$

$$6 B \frac{1 + \dfrac{t_2}{t_1}\dfrac{b_2}{b_1}}{t_1 b_1^2 b_2\left(1 + 2\dfrac{t_2}{t_1}\dfrac{b_2}{b_1}\right)} - \sigma_0 \leq 0. \tag{11}$$

The expression (10) represents the constraint function corresponding to the given stress constraints.

2 RESULTS AND DISCUSSION

2.1 Analytic Solution

The Lagrange multiplier method [6] to [7] and [21] to [22] is a powerful tool for solving these kind of problems and represents a classical approach to constraint optimization. Lagrange multiplier, labeled as λ, measures the change of

the objective function with respect to the constraint. The Lagrange multiplier method is a powerful tool for solving this class of problems without the need to explicitly solve the conditions and use them to eliminate extra variables. It is a method for finding the extremum of the function of several variables when the solution must satisfy a set of constraints, and for the analogous problem in the calculus of variations.

Applying the Lagrange multiplier method to the vector which depends on two parameters b_i ($i = 1, 2$) the system of equations

$$\frac{\partial}{\partial b_i}\left[A(b_1,b_2) + \lambda \varphi(b_1,b_2)\right] = 0, \quad i = 1, 2, \tag{12}$$

will be obtained and after the elimination of the multiplier λ, it will become

$$\frac{\partial A(b_1,b_2)}{\partial b_1} \cdot \frac{\partial \varphi(b_1,b_2)}{\partial b_2} = \frac{\partial A(b_1,b_2)}{\partial b_2} \cdot \frac{\partial \varphi(b_1,b_2)}{\partial b_1}. \tag{13}$$

Let the ratio

$$z = b_2 / b_1 \tag{14}$$

be the optimal ratio of the parts of the considered cross-section and let

$$\psi = t_2 / t_1, \tag{15}$$

be the ratio of the flange and web thicknesses.

After the introduction of the Eq. (6) for the bimoment into the Eq. (11), the Eq. (13) can be reduced to an equation whose solutions give the optimal values of the ratio (14). The solutions are in the form of the sixth order

$$\sum_{k=1}^{6} c_k z^k = 0. \tag{16}$$

The coefficients c_k in (16) are defined as:

$$c_0 = 60 + 144(\xi_1 + \xi_2),$$

$$c_1 = 2\psi(145 + 276\xi_1 + 264\xi_2) + 36(3\xi_1 + 4\xi_2)\frac{M_y}{M_x},$$

$$c_2 = 2\psi\left\{ \begin{matrix} (55 + 38\xi_1 + 27\xi_2) \\ -3[4 - 23(3\xi_1 + 4\xi_2)]\frac{M_y}{M_x} \end{matrix} \right\},$$

$$c_3 = \psi^2 \left\{ \begin{matrix} 4\psi(50 - 75\xi_1 - 84\xi_2) \\ -[123 - 76(3\xi_1 + 4\xi_2)]\frac{M_y}{M_x} \end{matrix} \right\},$$

$$c_4 = -\psi^3 \left\{ \begin{matrix} 8\psi(31\xi_1 + 24\xi_2) \\ +[208 + 75(3\xi_1 + 4\xi_2)]\frac{M_y}{M_x} \end{matrix} \right\},$$

$$c_5 = -2\psi^4 \left[16\psi\xi_1 + 31(2 + 3\xi_1 + 4\xi_2)\frac{M_y}{M_x} \right],$$

$$c_6 = -8\psi^5(2 + 3\xi_1 + 4\xi_2)\frac{M_y}{M_x}. \tag{17}$$

It is obvious that the coefficients c_k ($k = 1, 2, …, 6$) depend on the ratio of the bending moments M_y / M_x and on the eccentricities ξ_1 and ξ_2 of their planes.

The results that follow were obtained by an analytical approach.

2.2 Optimal Values

From the general case, when bending moments about both axes appear simultaneously with the bimoment, some particular cases can be considered, depending on the ratio M_y / M_x.

The optimal ratios b_2 / b_1 defined by (14) and obtained from the Eq. (16) are calculated for $M_y / M_x = 0, 0.1, 0.5, 1$; $\psi = 0.5, 0.75, 1$ and $\xi_1, \xi_2 = 0, 0.2, 0.4, 0.6, 0.8, 1.0$, or in other way, for $0 \le \xi_1 \le 1$; $0 \le \xi_2 \le 1$.

The optimal values of z for $M_y / M_x = 0$ and $\psi = 0.5, 0.75$ and 1.0, are shown in Tables 1, 2 and 3 as the functions of ξ_1 and ξ_2.

The highest and the lowest optimal values $z = b_2 / b_1$ for $M_y / M_x = 0.1, 0.5, 1$ are given in a shortened form in Table 4.

From Tables 1 to 4 it is evident that the quantity z is decreasing when the eccentricities ξ_1 and ξ_2 are increasing. Also, it can be concluded that the values of z are decreasing when the ratio $\psi = t_2 / t_1$ is increasing and that they are decreasing when the load ratio is increasing.

Table 1. *Optimal z for $M_y/M_x = 0$, $\psi = 0.5$*

$\xi_1 \rightarrow$ $\xi_2 \downarrow$	0	0.2	0.4	0.6	0.8	1
0	8.99	6.33	4.27	3.53	3.15	2.92
0.2	7.55	4.62	3.66	3.21	2.96	2.79
0.4	5.20	3.82	3.29	2.99	2.82	2.69
0.6	4.03	3.38	3.05	2.85	2.71	2.62
0.8	3.48	3.10	2.88	2.73	2.63	2.55
1	3.16	2.92	2.76	2.65	2.57	2.50

Table 2. *Optimal z for $M_y/M_x = 0$, $\psi = 0.75$*

$\xi_1 \rightarrow$ $\xi_2 \downarrow$	0	0.2	0.4	0.6	0.8	1
0	6.04	4.22	2.85	2.35	2.1	1.95
0.2	5.99	3.08	2.44	2.14	1.97	1.86
0.4	3.47	2.55	2.19	2.00	1.88	1.80
0.6	2.69	2.25	2.03	1.90	1.81	1.74
0.8	2.32	2.07	1.92	1.82	1.75	1.70
1	2.11	1.94	1.84	1.77	1.71	1.67

Table 3. *Optimal z for $M_y/M_x = 0$, $\psi = 1$*

$\xi_1 \rightarrow$ $\xi_2 \downarrow$	0	0.2	0.4	0.6	0.8	1
0	4.53	3.17	2.14	1.77	1.58	1.46
0.2	4.49	2.31	1.83	1.61	1.48	1.40
0.4	2.60	1.91	1.64	1.50	1.41	1.35
0.6	2.02	1.69	1.52	1.42	1.36	1.31
0.8	1.74	1.55	1.44	1.37	1.32	1.28
1	1.58	1.46	1.38	1.32	1.28	1.25

Table 4. *Optimal $z = b_2/b_1$ for $M_y/M_x = 0.1, 0.5$, 1 and $\psi = 0.5; 0.75; 1$*

M_y/M_x	ψ	z
0.1	0.5	$2.61 \leq z \leq 5.54$
	0.75	$1.75 \leq z \leq 4.38$
	1	$1.31 \leq z \leq 3.70$
0.5	0.5	$2.53 \leq z \leq 2.74$
	0.75	$1.70 \leq z \leq 2.19$
	1	$1.28 \leq z \leq 1.86$
1	0.5	$2 \leq z \leq 2.49$
	0.75	$1.60 \leq z \leq 1.67$
	1	$1.26 \leq z \leq 1.37$

2.3 The Loading Cases

The obtained results are used for the calculation that follows. Some particular cases can be considered, depending on the loading case. In the present section, a cantilever Z-beam is fixed at one end and subjected to the concentrated bending moments $M_x = 100$ Nm, $M_y = 0$ Nm ($M_y/M_x = 0$).

The loading cases were considered when concentrated bending moments were applied at the free end for three positions of the load plane with respect to the shearing plane (Fig. 2). The optimal values z_{opt} are calculated as previously explained.

3 NUMERICAL EXAMPLE AND ANALYSIS OF RESULTS

As a numerical example, the considered cantilever beam with the length $l = 1500$ mm, fixed at one end, is subjected to the bending moments $M_x = 100$ Nm, $M_y = 0$.

The initial cross-sectional geometrical characteristics are calculated taking into account the initial dimensions of the Z-section beam. It is assumed that the considered section has the initial cross-sectional geometrical characteristics: $b_1 = 51.75$ mm, $b_2 = 92$ mm, $t_1 = 8$ mm, $t_2 = 6.5$ mm. It represents the **Initial model** with the "Initial area" of the cross-section. For the given loads (Figs. 3 and 4) and the defined geometry of the profile, the initial stresses are calculated.

Starting from the initial relation $z_{initial}$ and for the initial wall thicknesses t_1 and t_2 the optimal relation $z_{optimal}$ is calculated defining the "Optimal area" of the cross-section.

3.1 Determination of the Minimum Cross Sectional Area

The problem is considered in two ways:
1) The optimal dimensions of the cross-section $b_{1optimal}$ and $b_{2optimal}$ are obtained by equalizing the "Initial" and the "Optimal area" ($A_{initial} = A_{optimal}$) and by using the calculated optimal relation z. In that case the normal stress lower than the initial one is obtained ($\sigma_{optimal} < \sigma_{initial}$). It represents the **Optimal model no. 1** (Table 5).
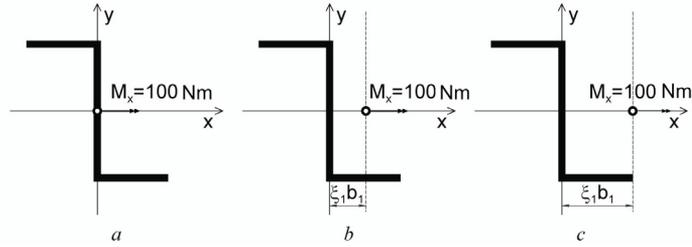
Fig. 2. *a) Loading case 1: $\xi_1 = 0$, $\xi_2 = 0$, b) Loading case 2: $\xi_1 = 0.5$, $\xi_2 = 0$, c) Loading case 3: $\xi_1 = 1$, $\xi_2 = 0$*

Table 5. *Normal stresses and saved mass: $t_1 = 0.8$ cm and $t_2 = 0.65$ cm, $z_{initial} = 1.78$*

Loading Case	$z_{optimal}$	$\sigma_{initial}$ [MPa]	$\sigma_{optimal\,no.1}$ [MPa]	$\sigma_{optimal\,no.2}$ [MPa]	$A_{initial} = A_{optimal\,no.1}$ [mm^2]	$A_{min} = A_{optimal\,no.2}$ [mm^2]	Saved mass [%]
1	5.58	13.4	7.5	13.4	1426	1033	27.56
2	4.18	15.9	12.7	15.9	1426	1398	1.96
3	1.54	38.2	37.9	38.2	1426	1418	0.56

2) From the condition prescribing that the stresses must be lower than the allowable one, i.e. the "Initial stress", the optimal values $b_{1optimal}$ and $b_{2optimal}$ are obtained by using the calculated optimal relation z and by comparing the stress defined by the optimal geometrical characteristics to the "Initial stress". It represents the **Optimal model no. 2**. Starting from the optimal cross sectional dimensions ($b_{1optimal}$ and $b_{2optimal}$), the optimal – minimum cross-sectional area A_{min} is calculated for each loading case and the results including the saved mass of the material are given in Table 5. From the Table 5 it can be seen that for all the loading cases the level of stresses is decreased in the Optimal model no. 1 with the area of the cross-section having the same value as in the Initial model, and the saved mass of material is increased with respect to the initial stress limits in the Optimal model no. 2 where the area is smaller than the initial one. The calculation showed that the maximum saved material is obtained in the Loading case 1 and the minimum in the Loading case 3. This allows the conclusion that if the distance of the loading plane from the shearing plane is increased, the optimization of the cross-section is less necessary.

4 CONCLUSIONS

The paper presents one approach to the optimization of the thin-walled Z – section beams, loaded in a complex way, using the Lagrange multiplier method.

Accepting the cross-sectional area as the objective function and the stress constraints as the constrained functions, it is possible to calculate the optimal ratios of the webs and the flanges of the considered thin-walled profiles in a very simple way.

In addition to the general case, some particular loading cases are considered. As a result of the calculation the modified constrained functions are derived as the polynomials of the sixth order.

Particular attention is directed to the calculation of the saved mass using the proposed analytical approach. It is also possible to calculate the saved mass of the used material for different loading cases.

The aim of the paper is the optimization of thin-walled elements subjected to the complex loads, and it may be concluded that the paper gives the general results permitting the derivation of the expressions that can be recommended for technical applications.

5 REFERENCES

[1] Timoshenko, S.P., Gere, J.M. Theory of elastic stability, 2nd ed. Mc Graw-Hill, New York, 1961, ISBN-10 0070647496.

[2] Vlasov, V.Z. *Thin-Walled Elastic Beams*, 2nd ed., Moscow, 1959, p. 568. (English translation, Israel Program for Scientific Translation, Jerusalem, 1961).

[3] Kollbruner, C.F., Hajdin, N. Dunnwandige Stabe, Band 1, *Springer Verlag*, Berlin, 1972.

[4] Kollbruner, C.F., Hajdin, N. Dunnwandige Stabe, Band 2, *Springer Verlag*, Berlin, 1975.

[5] Murray, N.W. Introduction to the theory of thin-walled structures, *Clarendon Press*, Oxford, 1984, ISBN 0-19-856186-5.

[6] Rhodes, J., Spence, J. Behaviour of thin-walled structures, *Elsevier Applied Science*, London, 1984, ISBN: 0-85334-246-6.

[7] Fox, R.L. Optimization methods for engineering design, *Addison-Wesley Publishing Company Inc*, Reading, Massachusetts, 1971.

[8] Brousse, P. Structural optimization, *CISM, No. 237*, *Springer-Verlag*, Wien, 1975, ISBN 3-211-81376-4.

[9] Prager, W. Introduction to structural optimization, *CISM, No. 212*, *Springer-Verlag*, Wien, 1974, ISBN 978-3-211-81291-4.

[10] Rozvany, G. Shape and layout optimization of structural systems and optimality criteria methods, *CISM, No. 325*, *Springer-Verlag*, Wien, 1992, ISBN 978-3-211-82363-7.

[11] Deng,Y., Zheng, D. Minimizing the thicknesses of injection-molded plastic parts based on a moldflow simulation, *Strojniški vestnik - Journal of Mechanical Engineering,* 2007, vol. 53, no. 7-8, p. 503-518.

[12] Paczelt, I., Baksa, A., Szabo, T. Product design using a contact-optimization technique, *Strojniški vestnik - Journal of Mechanical Engineering,* 2007, vol. 53, no. 7-8, p. 442-461.

[13] Tian,Y.S., Lu, T.J. Minimum weight of cold-formed steel sections under compression, *Thin-walled structures,* 2004, vol. 42, no. 4, p. 515-532.

[14] Farkas, J. Optimum design of metal structures, *Akademiai KIADO*, Budapest, 1984.

[15] Selmic, R., Cvetkovic, P., Mijailovic, R., Kastratovic G. Optimum dimenzions of triangular cross-section in lattice structures, *Meccanica,* 2006, vol. 41, no. 4, p. 391-406.

[16] Ružić, D., Anđelić, N., Milošević V. One more view on the problem of optimization of an I-beam, *Bulletins for Applied Mathematics*, 1996, LXXIX, PC-116, God, TU Budapest, Hungary, p. 367-372.

[17] Andjelić, N. Thin walled I-beam under complex loads - Optimization according to stress constraint, *FME Transactions,* 2003, vol. 31, no. 2, p. 55-60.

[18] Andjelić, N., Milosević-Mitić V. Optimization of a thin-walled cantilever beam at constrained torsion, *Structural integrity and life,* 2006, vol. 6, no. 3, p. 121-128.

[19] Ružić, D. Strength of structures, *University of Belgrade, Faculty of Mechanical Engineering*, Belgrade, 1995, ISBN 86-7083-266-6, (in Serbian).

[20] Bojanić, Z. Theory of elasticity 2002, vol. 2, *University of Belgrade, Faculty of Mechanical Engineering*, Belgrade, ISBN 86-7083-416-2, (in Serbian),.

[21] Onwubiko, C. Introduction to engineering design optimization, *Prentice Hall, Inc*, New Jersey, 2000, ISBN 0-201-47673-8.

[22] Zoller, K. For precise interpretation of the Lagrange equation of the second type, *Ingenieur-Archiv,* 1972, vol. 41, no. 4, p. 270-277, (In German).