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Numerična dinamika tekočin Computational Fluid Dynamics

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Članek obravnava numerično reševanje prenosnih pojavov v tekočinah. Predstavljena robno-območna Integralna metoda ponuja nekatere prednosti v primerjavi z drugimi metodami (metodo končnih razlik in metodo končnih elementov). Prednosti izhajajo iz uporabe različnih Greenovih funkcij glede na vrsto obravnavanega problema. Na tej osnovi so izpeljane različne numerične sheme, med katerimi je najobetavnejša shema z Greenovimi funkcijami difuzivno-konvektivne enačbe, ki je stabilna ne glede na vrednost Pecletovega oziroma Reynoldsovega števila. Uporaba tehnike podobmočij ter modernih iterativnih metod omogoča močno zmanjšanje potreb metode po računalniškem času in spominu.

The paper deals with numerical solution of transport phenomena in fluids. Presented boundary-domain integral method offers some advantages in comparison with other domain-type methods (finite differences or finite elements methods). Advantages arise from application of different Green's functions depending on the type of problem under consideration. On this basis different numerical schemes are developed, among which the most promising is the scheme with Green's functions of diffusive-convective equation, which is stable regardless of Peclet or Reynolds number values. Application of subdomain technique and modern iterative methods enables great reduction of computer time and memory demands of the method.

0. UVOD

0. INTRODUCTION

Izredno hiter napredek računalništva je omogočil razvoj posebne veje numerične dinamike tekočin [1], [17]. Ta predstavlja numerično modeliranje in simuliranje tokovnih razmer ali tudi numerični preizkus, krmiljen z računalnikom. Numeričnemu preizkusu lahko pripišemo pomembne prednosti nasproti fizikalnemu preizkusu, krmiljenemu na laboratorijskih modelih. Velike prednosti so predvsem v tem, da lahko lastnosti tekočine (gostoto, viskoznost, stisljivost, itn.) preprosto in poljubno spremenimo, numerični preizkus ne moti toka, simuliramo ravninske tokove, kar je skoraj nemogoče doseči s fizikalnim preizkusom. Seveda pa ima numerični preizkus tudi pomanjkljivosti, lastne vsem numeričnim postopkom, saj pomeni numerična rešitev vedno rezultat diskretnega sistema enačb, ki niso analogne temeljnim fizikalnim zakonom mehanike zveznih teles. Diskretizacija pogosto spremeni kolikostno in kakovostno obnašanje enačb in rešitev. Numerično simuliranje ima tudi podobno omejitve kakor fizikalni laboratorijski preizkus, saj so rešitve le posamične diskrette vrednosti in ne funkcije odvisnosti tokovnega polja.

The very fast developments of computing have enabled the development of numerical fluid dynamics [1], [17], i.e. numerical modelling and simulation of flow circumstances, including numerical experiments, by a computer. Such a procedure may have several important advantages over physical measurements on a laboratory model. It is of great importance that fluid properties (density, viscosity, compressibility, etc.) can be simply and arbitrarily changed, a numerical experiment does not disturb the flow, and plane flows can simply be simulated, which may not be the case with laboratory experiments. A numerical experiment also has its own drawbacks and disadvantages, as with all numerical procedures, since a numerical solution is the result of discrete equation systems, which are not completely identical to basic physical laws of the mechanics of continua. Discretisation often changes the behaviour of equations quantitatively and qualitatively and thus also the solutions. Numerical simulation has also similar limitations to laboratory experiments, since the solutions are individual discrete values only, not the functions of the flow fields.

Čeprav se je numerična dinamika tekočin potrdila kot izviren način študija tokovnih razmer, vendar ne more v celoti nadomestiti fizikalnega preizkusa in teoretične analize. Zaradi izjemne težavnosti predmeta so vsi trije omenjeni načini analize enakovredni in nujno potrebeni. Dinamika tekočin je področje raziskovanja, izrazito bogatega z nelinearnostmi, močnimi vplivi geometrijskih nepravilnosti in singularnih robnih pogojev. Vodilne enačbe prenosnih pojavov so v splošnem difuzivno-konvektivne parcialne diferencialne enačbe, katerih značaj se močno spreminja od točke do točke tokovnega polja, kar je odvisno od vrednosti lokalnih Reynoldsovih ozjroma Pecletovih števil, ki fizikalno pomenijo razmerje med difuzijo in konvekcijo določene veličine. Tako ne moremo govoriti o čistih eliptičnih, paraboličnih in hiperboličnih enačbah, ampak o enačbah mešanega tipa. Prav ta mešani značaj enačb dela numerično dinamiko tekočin neprimerno težjo od numeričnega reševanja pojavov v trdni snovi.

Navier-Stokesove enačbe so sistem nelinearnih parcialnih diferencialnih enačb gibanja viskozne newtonske tekočine. Sistem je matematični model osnovnih fizikalnih zakonov ohranitve mase, energije, snovi in gibalne količine, veljaven za nadzorno prostornino – Eulerjev način. Vodilne enačbe lahko zapišemo za osnovne fizikalne spremenljivke ali tudi za izpeljane. Na izbiro najprimernejšega oblikovanja v veliki meri vpliva uporabljeni numerična tehnika. Tako poznamo hitrostno-tlačno, vrtinčno-tokovno, hitrostno-vrtinčno, »penalty« izražanje itn. Zlasti hitrostno-vrtinčni način se je pokazal kot zelo ugoden pri metodi robnih elementov. Privlačnost hitrostno-vrtinčnega izražanja je predvsem v numerični ločitvi kinematike in kinetike toka od računanja tlaka. Tlak izračunamo pozneje z rešitvijo linearnega sistema enačb že znanega hitrostnega in vrtinčnega polja.

Nelinearni sistem Navier-Stokesovih enačb v splošnem popisuje tako laminaren kakor tudi turbulenten tok. Omenjene enačbe le izjemoma uporabljamo pri numeričnem simuliraju turbulentnih tokov pri večjih vrednostih Reynoldsovega števila, saj so te rešitve izredno drage, vezane na največje računalnike in za praktične inženirske probleme skoraj nemogoče. Pri reševanju inženirskeih turbulentnih problemov se zato moramo zadovoljiti z določenimi poenostavitvami na podlagi statističnih postopkov časovnega povprečja. Znani so številni načini reševanja turbulence, npr. popolno simuliranje turbulence (FTS), simuliranje velikih vrtljev (LES), Reynoldsovi časovno povprečni modeli itn. Popolno simuliranje je preprosto numerična rešitev Navier-Stokesovih enačb vseh detajlov turbulentnega toka. Takšno modeliranje je vedno prostorsko in časovno odvisno. Zapis z velikimi

Even though numerical fluid dynamics has been recognised as an original attempt to study flow circumstances, it cannot totally replace physical experiments and theoretical analyses. Due to the difficulties of the subject, all these approaches are equally important and essential. Fluid dynamics is a research field, full of nonlinearities, strong geometrical nonregularities and singularities due to boundary conditions. Governing equations of transport phenomena are in general diffusivity-conductivity partial differential equations, the characteristics greatly change from point to point in the flow field, due to local Reynolds and Pecllet number values, physically representing the relationship between diffusion and convection of individual parameters of state. Thus, it is not possible to discriminate pure elliptic, parabolic and hyperbolic equations, since they are of mixed type. This particular character of the equations makes numerical fluid dynamics more difficult compared to the numerical solving of phenomena in solids.

Navier-Stokes equations are a system of nonlinear partial differential equations of viscous Newtonian fluid motion. They are a mathematical model of physical conservation laws of mass, energy, species and momentum for a control volume – Eulerian case. Governing equations may be written for primitive physical variables or for dependent ones. For selection of the best formulation, it is of great importance which numerical technique is applied. There is a variety of velocity-pressure, vorticity-stream functions, velocity-vorticity, penalty formulations, etc. In particular, the velocity-vorticity approach has shown advantages over the boundary element method. The advantage of the velocity-vorticity formulation lies with the numerical separation of the kinematics and kinetics of the flow from the pressure computation, which is determined later by the solution of a linear system of equations for known velocity and vorticity fields.

Non-linear Navier-Stokes equations in general govern laminar as well as turbulent flow. These equations are used only exceptionally in numerical simulation of turbulent flows for larger Re number values, as they are extremely expensive and applicable only on real supercomputers, which makes computation of practical engineering problems virtually impossible. We are therefore forced to make some simplifications based on statistical means of time averaging if we want to calculate practical engineering turbulence problems. There are many known approaches to turbulence modelling, including full turbulence simulation (FTS), large eddy simulation (LES), Reynolds time-averaged models etc. FTS is simply a numerical solution of Navier-Stokes equations for all details

vrtlji je prav tako prostorsko in časovno odvisen, eksplisitno simuliramo le velike vrtlje, medtem ko manjše modeliramo. Kljub izrednemu razvoju računalništva sta načina FTS in LES praktično predraga za reševanje inženirskega primerov turbulence. Namenjena pa sta lahko za študij turbulence. Reševanje časovno povprečnih Navier-Stokesovih enačb prek Reynoldsove razvezitve trenutnih veličin toka na časovno povprečne vrednosti in trenutne deviacije oziloma fluktacije od časovno povprečne vrednosti je še vedno najpogosteje uporabljan način v numeričnem simuliranju turbulentnih tokov.

Pri mnogih praktičnih aproksimacijah nas navadno zanima le ustaljeno stanje, manj pa sam prehoden pojav. Vsem numeričnim tehnikam je skupno, da najuspešnejša tokovna simuliranja tudi ustaljenega stanja, srujejo na časovno odvisnih enačbah, rezultate ustaljenega stanja pa dobimo s časovnim omejevanjem prehodnih rešitev. Časovno odvisen sistem enačb je numerično laže obvladljiv, je bolj stabilen, saj je znotraj posameznih časovnih intervalov bistveno manj nelinearen. Prehoden način tudi ne predpostavlja ustaljenega stanja, ki lahko v resnici tudi ne obstaja.

1. OSNOVE DINAMIKE TEKOČIN

1.1 Zakoni ohranitve

Sistem vodilnih parcialnih diferencialnih enačb prenosnih pojavov v toku nestisljive tekočine podamo z osnovnimi fizikalnimi zakoni ohranitve mase, globalne količine, energije in snovi, zapisan npr. v kartezijevem tenzorskem zapisu [3]:

of turbulent flow. Such modelling is always space and time dependent. LES is also a space and time dependent approach, where only large eddies are explicitly simulated, whereas little ones are modelled. In spite of the remarkable development of modern computers, FTS and LES are still in practice too expensive for solving real engineering problems, but they can serve well as an efficient tool for studying turbulence. Solving a time-averaged form of Navier-Stokes equations with Reynolds decomposition of instantaneous flow quantities into time-averaged part and instantaneous deviations (or fluctuations) from time-averaged values is still the most common way of simulating turbulent flows.

With most practical approximations, only the steady state is of interest with respect to transient phenomena. It is common to all numerical techniques that they are usually more effective, also when determining the steady state, if this is achieved from time dependent solutions by a limit process. A time dependent system of equations is numerically simpler to deal with, and it is more stable since it has less nonlinear behaviour in individual time steps. Transient case approach also does not presume the existence of a steady state, which may not always exist.

1. BASIC FLUID DYNAMICS

1.1 Conservation Laws

The partial differential equations set governing transport phenomena in incompressible fluid flow represent the basic conservation balances of mass, momentum, energy and species concentration written below in an indicial notation form for a right-handed Cartesian coordinate system [3]:

$$\frac{\partial v_j}{\partial x_j} = 0 \quad (1.1)$$

$$\rho_0 \frac{D v_i}{D t} = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho_0 f_{m_i} \quad (1.2)$$

$$\rho_0 c_p \frac{DT}{Dt} = -\frac{\partial q_{Tj}}{\partial x_j} \pm I_T + \Phi \quad (1.3)$$

$$\frac{DC}{Dt} = -\frac{\partial q_{Cj}}{\partial x_j} \pm I_C \quad (1.4)$$

za $I, j = 1, 2, 3$, kjer pomenijo: v_i – lokalno trenutno hitrost v x_i koordinatnih smereh, D/Dt – snovni odvod oziloma Stokesov odvod, σ_{ij} – napetostni tenzor in f_{m_i} – prostorninske sile (npr.: gravitacija g_i), ρ_0 , c_p , I_T in I_C – konstantna gostota tekočine, specifična izobarna toplota, toplotni vir oz. ponor toplote in snovi zaradi kemijske reakcije, T – temperatura, C – koncentracija, q_{Tj} in q_{Cj} – gostoti toplotnega toka oziloma toka snovi. Člen Φ je Rayleighova viskozna trosilna funkcija podana z:

for $I, j = 1, 2, 3$, where v_i is the i-th instantaneous velocity component, x_i is the i-th coordinate, D/Dt represents the substantial or Stokes derivative, σ_{ij} is the stress tensor and f_{m_i} stands for the body force, e.g. the gravity g_i , ρ_0 , c_p , I_T and I_C are respectively the constant fluid mass density, specific isobaric heat capacity, heat source or sink, and the rate of production or destruction of species by chemical reaction. T is the temperature and C the concentration, while q_{Tj} and q_{Cj} are heat and mass fluxes respectively. The term Φ is the Rayleigh viscous dissipation function, given by:

$$\Phi = \tau_{ij} \frac{\partial v_i}{\partial x_j} \quad (1.5)$$

ki spreminja razpoložljivo mehansko energijo v toplotno in deluje kot dodatni toplotni vir, s τ_{ij} viskoznim napetostnim tenzorjem:

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij} \quad (1.6)$$

kjer sta p – trenutni tlak, δ_{ij} – Kroneckerjeva funkcija delta. Vektor gostote toplotnega toka \vec{q}_T je v splošnem primeru vsota difuzijskega \vec{q}_{TD} in sevalnega toka \vec{q}_{TR} :

$$\vec{q}_T = \vec{q}_{TD} + \vec{q}_{TR} \quad (1.7)$$

Pri reševanju praktičnih problemov neizotermnih tokov moramo upoštevati v gibalni enačbi (1.2) vzgonske sile, saj lahko pomembno vplivajo na razvoj hitrostnega, temperaturnega in koncentracijskega polja [23], [40]. Te lahko vključimo v analizo z Boussinesqovo aproksimacijo, kjer predpostavimo spremenljivost gostote le pri prostorninskih silah \vec{f}_m , pri vseh drugih členih enačbe pa zanemarimo spremenljivost gostote in jo imamo za konstantno. S substitucijo funkcijске odvisnosti za gostoto splošne oblike:

$$\rho = \rho_0 (1 + F) \quad (1.8)$$

lahko enačbo (1.2) zapišemo v obliki:

$$\rho_0 \frac{Dv_i}{Dt} = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho_0 (1 + F) g_i \quad (1.9)$$

Pri izbiri zakonitosti F moramo biti pazljivi, saj neustrezen zakon vodi do napačne rešitve hitrostnega in temperaturnega polja. Pogosto zadošča linearna zakonitost spremenjanja gostote v odvisnosti od temperature, npr. za zrak, olje itn., v normalizirani obliki razlike gostote:

$$\frac{\rho - \rho_0}{\rho_0} = F = -\beta_T (T - T_0) \quad (1.10)$$

kjer sta β_T – toplotni prostorninski razteznostni koeficient, ρ_0 – referenčna gostota pri temperaturi T_0 . Za nekatere tekočine moramo upoštevati nelinearni zakon spremenjanja gostote s temperaturo, npr. kvadratno funkcijo za vodo v bližini točke anomalije [42]. V nekaterih posebnih primerih procesnega inženirstva moramo upoštevati hkratni vpliv temperaturnega in koncentracijskega gradienta na gostoto tekočine. Sem sodijo pojavi pri sušenju, sončnih toplotnih zbiralnikih, zbiralnikih utekočinjenega naravnega plina, atmosferski tokovi itn. Funkcijsko odvisnost med gostoto, temperaturo in koncentracijo predstavlja na primer naslednji izraz [41]:

$$\frac{\rho - \rho_0}{\rho_0} = F = -[\beta_T (T - T_0) + \beta_C (C - C_0)] \quad (1.11)$$

kjer sta β_C – prostorninski razteznostni koeficient in C_0 – referenčna koncentracija pri temperaturi T_0 .

which converts the mechanical energy to heat acting as an additional heat source term, with τ_{ij} the viscous stress tensor:

where p is the instantaneous pressure and δ_{ij} the Kronecker delta function. The heat flux vector \vec{q}_T is in general case a sum of the diffusive flux \vec{q}_{TD} and the radiation flux \vec{q}_{TR} :

$$\vec{q}_T = \vec{q}_{TD} + \vec{q}_{TR} \quad (1.7)$$

In many nonisothermal engineering applications, buoyancy forces play an important role in developing velocity, temperature and concentration fields, and they thus have to be included in the momentum conservation eq. (1.2) [23], [40]. This can be accomplished by using a Boussinesq approximation, where the temperature influence on fluid mass density is considered only with the body force term \vec{f}_m , while it is neglected in all other terms, where the fluid mass density is considered to be invariant. Substituting the expression for density variation of a general form:

(1.2) can be written in the following form:

The function F has to be chosen physically justified, otherwise the results for its velocity and temperature fields may be unrealistic. For a number of fluids, e.g. air, oil etc., a simple linear approximation of fluid density temperature dependence is sufficient, given by a normalised difference density expression:

where ρ_0 is the reference density at temperature T_0 , and β_T is the volume coefficient of thermal expansion. For some fluids, nonlinear approximations of density temperature variations are necessary to account for the material behaviour, e.g. a quadratic function has to be used for water around the point of anomaly [42]. In some special cases of process engineering, the simultaneous effect of temperature and concentration fields on density variation has to be considered, e.g. by the following statement [41]:

where β_C is a volume coefficient of the concentration expansion and C_0 is the reference concentration at temperature T_0 .

1.2 Kinematika

Podajmo nekaj osnovnih vektorskih in tenzorskih veličin hitrostnega vektorskogopolja $\vec{v} = \vec{v}(\vec{r}, t)$ s kartezijivimi komponentami $\vec{v} = (v_x, v_y, v_z)$, kjer je $\vec{r} = (x, y, z)$ vektor legi [3]. Definirajmo tenzor hitrostnega gradijenta $\partial v_i / \partial x_j$ ozziroma $\vec{\nabla} \vec{v}$, ali grad \vec{v} v simbolnem zapisu z enakostjo:

$$\frac{\partial v_i}{\partial x_j} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) + \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right) \quad (1.12)$$

simetričnega tenzora $\dot{\epsilon}_{ij}$ deformacijskih hitrostih:

$$\dot{\epsilon}_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \quad (1.13)$$

in antisimetričnega tenzora $\dot{\Omega}_{ij}$ kotnih hitrosti:

$$\dot{\Omega}_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right) \quad (1.14)$$

medtem ko deformacijsko hitrost podajmo z izrazom:

$$\dot{\gamma} = \sqrt{2\dot{\epsilon}_{ij}\dot{\epsilon}_{ij}} \quad (1.15)$$

Vektor vrtinčnosti $\vec{\omega}(\vec{r}, t)$ je rotor hitrostnega polja $\vec{\nabla} \times \vec{v}$ ali rot \vec{v} , v simbolnem zapisu:

$$\omega_i = e_{ijk} \frac{\partial v_k}{\partial x_j} \quad (1.16)$$

velja pa tudi naslednja pomembna odvisnost:

$$\vec{\omega} = 2 \vec{\Omega} \quad (1.17)$$

ozziroma fizikalna kotna hitrost $\vec{\Omega}(\vec{r}, t)$ infinitezimalnega delca tekočine je enaka polovici vektorja vrtinčnosti $\vec{\omega}$. V (1.16) je e_{ijk} permutacijski enotski tenzor, ki je enak 1 za krožni potek indeksov $i j k$, npr. 12312 ozziroma je enak -1 za protikrožni potek indeksov 32132 in je enak nič za vsako ponovitev indeksov. Ker pa velja za vsak vektor zveza $\operatorname{div} \operatorname{rot} \vec{v} = 0$, je vektor vrtinčnosti solenoidal vektor $\operatorname{div} \vec{\omega} = 0$:

$$\frac{\partial \omega_i}{\partial x_i} = 0 \quad (1.18)$$

Vrtinčnost $\vec{\omega}$ in antisimetrični tenzor $\dot{\Omega}_{ij}$ povezuje enačba:

$$\omega_k = -2e_{ijk}\dot{\Omega}_{ij} \quad (1.19)$$

ozziroma obratna zveza:

$$\dot{\Omega}_{ij} = -\frac{1}{2}e_{ijk}\omega_k \quad (1.20)$$

Stokesov odvod hitrostnega polja izraža pospešek fluidnega delca:

$$\frac{D v_i}{D t} = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = \frac{\partial v_i}{\partial t} + v_j \left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right) + v_j \frac{\partial v_j}{\partial x_i} \quad (1.21)$$

1.2 Kinematics

Let us define some necessary vector and tensor quantities of the velocity vector field $\vec{v} = (\vec{r}, t)$, with the cartesian components $\vec{v} = (v_x, v_y, v_z)$ and where $\vec{r} = (x, y, z)$ is the position vector [3]. The velocity gradient $\partial v_i / \partial x_j$ or $\vec{\nabla} \vec{v}$, and in symbolic notation $\operatorname{grad} \vec{v}$, is a tensor given with the identity:

of the symmetric tensor $\dot{\epsilon}_{ij}$ of deformat. velocities:

$$\dot{\epsilon}_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \quad (1.13)$$

and the antisymmetric tensor $\dot{\Omega}_{ij}$ of rotation velocities:

$$\dot{\Omega}_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right) \quad (1.14)$$

while the deformation velocity is given by the definition:

$$\dot{\gamma} = \sqrt{2\dot{\epsilon}_{ij}\dot{\epsilon}_{ij}} \quad (1.15)$$

The vorticity vector $\vec{\omega}(\vec{r}, t)$ is a curl of the velocity field, $\vec{\nabla} \times \vec{v}$ or $\operatorname{rot} \vec{v}$, in symbolic notation:

and the following important relation may be written:

$$\vec{\omega} = 2 \vec{\Omega} \quad (1.17)$$

or the physical angular velocity $\vec{\Omega}(\vec{r}, t)$ of an infinitesimal fluid particle is equal to one-half the vorticity vector $\vec{\omega}$. Above e_{ijk} in (1.16) is the permutation unit tensor, which equals 1, if the subscripts $i j k$ are in cyclic order 12312, or equals -1 when they are in anticyclic order 32132 and is otherwise zero. The effect of the vector operation $\operatorname{div} \operatorname{rot} \vec{v} = 0$, which holds for any vector function, is that the vorticity vector is a solenoidal vector $\operatorname{div} \vec{\omega} = 0$:

Vorticity $\vec{\omega}$ and the antisymmetric tensor $\dot{\Omega}_{ij}$ are related by the expression:

or with the inverse relation:

The Stokes derivative of the velocity field expressing fluid particle acceleration:

ozziroma zaradi definicije (1.19):

$$\frac{Dv_i}{Dt} = \frac{\partial v_i}{\partial t} + 2v_j \dot{\Omega}_{ij} + \frac{1}{2} \frac{\partial(v_j v_j)}{\partial x_i} \quad (1.22)$$

z upoštevanjem enačbe (1.20) pa tudi v obliki:

$$\frac{Dv_i}{Dt} = \frac{\partial v_i}{\partial t} - v_j e_{ijk} \omega_k + \frac{1}{2} \frac{\partial(v_j v_j)}{\partial x_i} \quad (1.23)$$

1.3 Zakoni tečenja

Omejimo se na sevalno prosojne tekočine, za katere veljajo preproste konstitutivne hipoteze gradientnega tipa, npr. Newtonov, Fourierjev in Fickov zakon, ki pomenijo odvisnost med napetostnim tenzorjem τ_{ij} in deformacijskim tenzorjem $\dot{\epsilon}_{ij}$:

vektorjem toplotnega toka \vec{q}_T in temperaturnim poljem $T(\vec{r}, t)$:

$$\tau_{ij} = 2\eta \dot{\epsilon}_{ij} \quad (1.24)$$

In med vektorjem toka snovi \vec{q}_C in koncentracijskim poljem $C(\vec{r}, t)$:

$$q_{Cj} = -a_C \frac{\partial C}{\partial x_j} \quad (1.26)$$

kjer so η , λ in a_C – dinamična viskoznost tekočine, toplotna prevodnost in difuzivnost snovi. Viskozno disipacijsko funkcijo Φ (1.5) nestisljive newtonske viskoznosti tekočine (1.24) podamo z zvezo [15]:

$$\Phi = 2\eta \dot{\epsilon}_{ij} \dot{\epsilon}_{ij} = 2\eta [\dot{\epsilon}_{11}^2 + \dot{\epsilon}_{22}^2 + \dot{\epsilon}_{33}^2 + 2(\dot{\epsilon}_{12}^2 + \dot{\epsilon}_{23}^2 + \dot{\epsilon}_{31}^2)] \quad (1.27)$$

2. NAVIER-STOKESOVE ENAČBE

2.1 Zapis z osnovnimi spremenljivkami

Z upoštevanjem konstitutivnih hipotez (1.24) do (1.26) v osnovnih ohranitvenih zakonih (1.1) do (1.9) izpeljemo vodilni nelinearni sistem Navier-Stokesovih enačb prenosnih pojavov v toku newtonske nestisljive tekočine:

$$\frac{\partial v_j}{\partial x_j} = 0 \quad (2.1)$$

$$\frac{Dv_i}{Dt} = -\frac{1}{\rho_0} \frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j} \left[\nu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \right] + Fg_i \quad (2.2)$$

$$\rho_0 c_p \frac{DT}{Dt} = \frac{\partial}{\partial x_j} \left(\lambda \frac{\partial T}{\partial x_j} \right) \pm I_T + \Phi \quad (2.3)$$

$$\frac{DC}{Dt} = \frac{\partial}{\partial x_j} \left(a_C \frac{\partial C}{\partial x_j} \right) \pm I_C \quad (2.4)$$

can be expressed using (1.19) as:

or due to the definition (1.20):

1.3 Constitutive Laws

Let us restrict the current discussion to a heat radiation transparent fluid, obeying a simple linear gradient type of constitutive hypothesis, namely Newton's, Fourier's and Fick's laws, describing respectively the relation between the stress tensor τ_{ij} and the strain tensor $\dot{\epsilon}_{ij}$:

$$\tau_{ij} = 2\eta \dot{\epsilon}_{ij} \quad (1.24)$$

the heat flux vector \vec{q}_T and the temperature field $T(\vec{r}, t)$:

$$q_{Tj} = -\lambda \frac{\partial T}{\partial x_j} \quad (1.25)$$

and between the dispersion flux vector \vec{q}_C and the concentration field $C(\vec{r}, t)$:

$$q_{Cj} = -a_C \frac{\partial C}{\partial x_j} \quad (1.26)$$

where η , λ and a_C are the respective fluid dynamic viscosity, heat conductivity, and mass diffusivity. The viscous dissipation function Φ (1.5) for incompressible fluids obeying the viscous Newton's law (1.24) is given by the relation [15]:

$$\Phi = 2\eta \dot{\epsilon}_{ij} \dot{\epsilon}_{ij} = 2\eta [\dot{\epsilon}_{11}^2 + \dot{\epsilon}_{22}^2 + \dot{\epsilon}_{33}^2 + 2(\dot{\epsilon}_{12}^2 + \dot{\epsilon}_{23}^2 + \dot{\epsilon}_{31}^2)] \quad (1.27)$$

2. NAVIER-STOKES EQUATIONS

2.1 Primitive Variables Formulation

Substituting the constitutive hypothesis eqs. (1.24) to (1.26) into the basic conservation laws, equations (1.1) to (1.9), the nonlinear Navier-Stokes equations set can be derived, expressing the transport phenomena in an incompressible Newtonian fluid flow:

kjer sta $\nu = \eta/\rho$ – kinematična viskoznost in $P = p - \rho_0 q_j r_j$ – modificiran tlak. Če predpostavimo, da so lastnosti snovi konstantne, kar je razumna predpostavka v številnih inženirskeih problemih, se Navier-Stokesove enačbe pomembno poenostavijo. Z upoštevanjem enakosti:

$$\frac{\partial}{\partial x_j} \left[\nu_0 \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \right] = \nu_0 \left[\frac{\partial^2 v_i}{\partial x_j \partial x_j} + \frac{\partial}{\partial x_i} \left(\frac{\partial v_k}{\partial x_k} \right) \right] \quad (2.5)$$

in kontinuitetne enačbe (2.1), lahko zapišemo naslednji sistem enačb:

$$\frac{\partial v_j}{\partial x_j} = 0 \quad (2.6)$$

$$\frac{D v_i}{D t} = \frac{\partial v_i}{\partial t} + \frac{\partial v_j v_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \nu_0 \frac{\partial^2 v_i}{\partial x_j \partial x_j} + F g_i \quad (2.7)$$

$$\frac{D T}{D t} = \frac{\partial T}{\partial t} + \frac{\partial v_j T}{\partial x_j} = a_{T_0} \frac{\partial^2 T}{\partial x_j \partial x_j} \pm \frac{I_T}{\rho_0 c_{p_0}} + \frac{\Phi}{\rho_0 c_{p_0}} \quad (2.8)$$

$$\frac{D C}{D t} = \frac{\partial C}{\partial t} + \frac{\partial v_j C}{\partial x_j} = a_{C_0} \frac{\partial^2 C}{\partial x_j \partial x_j} \pm I_C \quad (2.9)$$

kjer pomeni $a_T = \lambda/\rho c_p$ – topotno difuzivnost. Ta sistem enačb za konstantne lastnosti snovi pomeni sklenjen sistem enačb za določitev hitrostnega $\vec{v}(\vec{r}, t)$, tlachnega $p(\vec{r}, t)$, temperaturnega $T(\vec{r}, t)$ in koncentracijskega $C(\vec{r}, t)$ polja ob upoštevanju primernih začetnih in robnih pogojev za hitrost, temperaturo in koncentracijo. Sistem Navier-Stokesovih enačb (2.6), do (2.9) pomeni paraboličen problem začetnih robnih vrednosti, zato mu moramo za popolnost matematičnega opisa toka tekočine dodati Dirichletove, Neumannove in Cauchyjeve robne pogoje ter začetne vrednosti, na primer za hitrostno vektorsko polje $\vec{v}(\vec{r}, t)$:

$$\begin{aligned} v_j &= \bar{v}_j & \text{na/on } \Gamma & \text{za/for } t > 0 \\ v_j &= \bar{v}_{j_0} & v \text{ in } \Omega & \text{pri/at } t = t_0 \end{aligned} \quad (2.10)$$

in skalarno temperaturno polje $T(\vec{r}, t)$:

$$\begin{aligned} T &= \bar{T} & \text{na/on } \Gamma_1 & \text{za/for } t > t_0 \\ q_{T_n} &= \bar{q}_{T_n} & \text{na/on } \Gamma_2 & \text{za/for } t > t_0 \\ q_{T_n} &= \alpha_T (T - T_f) & \text{na/on } \Gamma_3 & \text{za/for } t > t_0 \\ T &= \bar{T}_0 & v \text{ in } \Omega & \text{pri/at } t = t_0 \end{aligned} \quad (2.11)$$

kjer je α_T – topotna prestopnost in T_f – temperatura okolice ter skalarno koncentracijsko polje $C(\vec{r}, t)$:

$$\begin{aligned} C &= \bar{C} & \text{na/on } \Gamma_1 & \text{za/for } t > t_0 \\ q_{C_n} &= \bar{q}_{C_n} & \text{na/on } \Gamma_2 & \text{za/for } t > t_0 \\ q_{C_n} &= \alpha_c (C - C_f) & \text{na/on } \Gamma_3 & \text{za/for } t > t_0 \\ C &= \bar{C}_0 & v \text{ in } \Omega & \text{pri/at } t = t_0 \end{aligned} \quad (2.12)$$

kjer je α_c – prestopnost snovi, C_f – koncentracija snovi v okolici.

where $\nu = \eta/\rho$ is the kinematic viscosity and $P = p - \rho_0 q_j r_j$ the modified pressure. If we assume that the material properties are constant, which is a reasonable assumption in many engineering problems, the Navier-Stokes equations set simplifies considerably. Due to the Identity:

and the continuity eq. (2.1), the following set of equations can be written

$$\frac{\partial v_j}{\partial x_j} = 0 \quad (2.6)$$

$$\frac{D v_i}{D t} = \frac{\partial v_i}{\partial t} + \frac{\partial v_j v_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \nu_0 \frac{\partial^2 v_i}{\partial x_j \partial x_j} + F g_i \quad (2.7)$$

$$\frac{D T}{D t} = \frac{\partial T}{\partial t} + \frac{\partial v_j T}{\partial x_j} = a_{T_0} \frac{\partial^2 T}{\partial x_j \partial x_j} \pm \frac{I_T}{\rho_0 c_{p_0}} + \frac{\Phi}{\rho_0 c_{p_0}} \quad (2.8)$$

$$\frac{D C}{D t} = \frac{\partial C}{\partial t} + \frac{\partial v_j C}{\partial x_j} = a_{C_0} \frac{\partial^2 C}{\partial x_j \partial x_j} \pm I_C \quad (2.9)$$

where $a_T = \lambda/\rho c_p$ is the thermal diffusivity. The above equations set for constant material properties represents a closed system of equations for the determination of velocity $\vec{v}(\vec{r}, t)$, pressure $p(\vec{r}, t)$, temperature $T(\vec{r}, t)$ and concentration $C(\vec{r}, t)$ fields, subject to appropriate initial and boundary conditions of velocity, temperature and concentration. The Navier-Stokes equation set (2.6) to (2.9) represents a parabolic initial-boundary value problem, and the mathematical description of the fluid motion is completed by providing suitable Dirichlet, Neumann or Cauchy mixed type boundary conditions, and some initial conditions have also to be known, e.g. for the velocity vector field $\vec{v}(\vec{r}, t)$:

for the temperature scalar field $T(\vec{r}, t)$:

where α_T and T_f respectively are the heat transfer coefficient and ambient fluid temperature, while for the scalar concentration field $C(\vec{r}, t)$ one may prescribe:

α_c being the mass transfer coefficient and C_f the ambient fluid concentration.

2.2 Zapis »Penalty«

Pri zapisu »penalty« stavka ohranitve gibalne količine (2.7) tlačni člen $p(\vec{r}, t)$ aproksimiramo s šibko kompresibilno obliko kontinuitetne enačbe [9], [10]:

$$p = -P \frac{\partial v_j}{\partial x_j} \quad (2.13)$$

Pogoj nestisljivosti je s tem porušen in je zapisan v obliki utežitvene omejitvene enačbe, v kateri je nestisljivost zapolnjena do vnaprej znane ravni, odvisne od vrednosti utežitvenega parametra »penalty« P . Stavek (2.13) vstavimo v gibalno enačbo in eliminiramo tlak kot osnovno veličino računanja. Za vrednosti utežitvenega parametra $P \rightarrow \infty$ je seveda izpolnjen pogoj o končni vrednosti tlaka in nestisljivosti tekočine $\partial v_j / \partial x_j \rightarrow 0$. V numerični analizi izberemo glede na natančnost računalnika neko veliko končno vrednost, npr. $P = 10^5$ do 10^7 .

2.3 Hitrostno-vrtinčni zapis

Z vektorjem vrtinčnosti $\vec{\omega}(\vec{r}, t)$ razdelimo postopek računanja tokovnih razmer na kinematski in kinetski del [20], [23], [36], [37], [39].

Kinematika je zajeta v kontinuitetni enačbi (2.6) in definiciji vrtinčnosti (1.16) in pomeni odvisnosti med vrtinčnim poljem in hitrostnim poljem v danem trenutku. Z omejitvijo obravnave na nestisljivo tekočino, ko je hitrostno polje solenoidalno $\operatorname{div} \vec{v} = 0$, ga lahko podamo z rotorjem vektorskega potenciala $\vec{v} = \operatorname{rot} \vec{\psi}$ ki ga izberemo poljubno solenoidalno $\operatorname{div} \vec{\psi} = 0$ oziroma v simboličnem zapisu:

$$v_i = \epsilon_{ijk} \frac{\partial \psi_k}{\partial x_j}, \quad \frac{\partial \psi_j}{\partial x_j} = 0 \quad (2.14)$$

Z neposredno kombinacijo enačb (2.14) in (2.16) izpeljemo vektorsko eliptično Poissonovo enačbo za hitrostni vektorski potencial:

$$\frac{\partial^2 \psi_i}{\partial x_j \partial x_j} + \omega_i = 0 \quad (2.15)$$

Z upoštevanjem rotorja (2.15) lahko kinematiko podamo tudi v obliki vektorske eliptične Poissonove enačbe za vektor hitrosti:

$$\frac{\partial^2 v_i}{\partial x_j \partial x_j} + \epsilon_{ijk} \frac{\partial \omega_k}{\partial x_j} = 0 \quad (2.16)$$

Kinetika je podana z vrtinčno prenosno enačbo, ki jo izpeljemo tako, da poščemo rotor gibalne enačbe (2.7), in opisuje prerazporeditev vrtinčnosti v toku tekočine:

$$\frac{D \omega_i}{Dt} = \frac{\partial \omega_i}{\partial t} + \frac{\partial v_j \omega_i}{\partial x_j} = \frac{\partial \omega_j v_i}{\partial x_j} + \nu_0 \frac{\partial^2 \omega_i}{\partial x_j \partial x_j} + \epsilon_{ijk} g_k \frac{\partial F}{\partial x_j} \quad (2.17)$$

2.2 Penalty Function Formulation

In penalty function formulation of the momentum conservation statement (2.7), the pressure term $p(\vec{r}, t)$ is approximated by a weak compressible form of the continuity equation [9], [10]:

$$p = -P \frac{\partial v_j}{\partial x_j} \quad (2.13)$$

The incompressibility condition is violated and written as a penalised constraint equation in which incompressibility is satisfied up to a predetermined level given by the penalty parameter P . The statement (2.13) is substituted into the momentum equation, eliminating the pressure from the primary computation. Since the pressure has a finite value, the penalty parameter goes off limits $P \rightarrow \infty$, due to the incompressibility of the media $\partial v_j / \partial x_j \rightarrow 0$. In the numerical analysis, some large values for the penalty parameters have to be taken, depending on the computer tolerance, e.g. $P = 10^5$ do 10^7 .

2.3 Velocity-Vorticity Formulation

With vorticity vector $\vec{\omega}(\vec{r}, t)$ the fluid motion computation scheme is partitioned into its kinematic and kinetic aspects [20], [23], [36], [37], [39].

The kinematics is described by the continuity (2.6) and the vorticity definition (1.16), expressing the relationship between the vorticity field at any given instant of time and the velocity field at the same instant. Due to the limitation to an incompressible fluid, the velocity field is solenoidal $\operatorname{div} \vec{v} = 0$, and it may be represented by the curl of the vector potential $\vec{v} = \operatorname{rot} \vec{\psi}$, which may be selected arbitrarily to be solenoidal, $\operatorname{div} \vec{\psi} = 0$ in a symbolic notation:

$$v_i = \epsilon_{ijk} \frac{\partial \psi_k}{\partial x_j}, \quad \frac{\partial \psi_j}{\partial x_j} = 0 \quad (2.14)$$

Combining directly (2.14) and (2.16), the following vector elliptic Poisson's equation is derived for the vector potential:

By taking the curl of (2.15) the kinematics can be also formulated in the form of a vector elliptic Poisson's equation for the velocity vector:

The kinetics is governed by the vorticity transport equation obtained as a curl of the momentum (2.7), and describes the redistribution of the vorticity in fluid flow:

Iz enačbe (2.17) izhaja, da je celotna spremembra vrtinčnosti delca tekočine podana s Stokesovim odvodom na levi strani enačbe, odvisna od členov na desni strani enačbe, ki pomenijo deformacijo, viskozno difuzijo in vzgonske sile. Difuzijski člen je popolnoma analogen členu v stavku gibalne količine, ki podaja difuzijo gibalne količine. Deformacijski člen ima pomen, če se vektor hitrosti spreminja vzdolž vrtinčne linije. Pri ravnniskem toku ima vektor vrtinčnosti $\vec{\omega}$ samo eno komponento pravokotno na ravnino toka, tako da ga lahko predstavimo s skalarno veličino ω . Vrtinčno deformacijski člen je nič ($\vec{\omega} \cdot \vec{\nabla} \vec{v} = 0$), tako da se vektorska vrtinčna enačba skrči v skalarno:

$$\frac{D\omega}{Dt} = \frac{\partial\omega}{\partial t} + \frac{\partial v_j \omega}{\partial x_j} = \nu_0 \frac{\partial^2 \omega}{\partial x_j \partial x_j} + e_{ij} g_j \frac{\partial F}{\partial x_i} \quad (2.18)$$

kjer je e_{ij} ($i, j = 1, 2$) permutacijski simbol ($e_{12} = +1, e_{21} = -1, e_{11} = e_{22} = 0$).

Glavni vzrok obravnavane toka tekočine v obliki za vrtično porazdelitev je v tem, da je vektor vrtinčnosti $\vec{\omega}$ (\vec{r}, t), solenoiden vektor, oziroma ne more nastati ali izginiti v notranjosti homogenega sredstva pri normalnih pogojih. Nastane samo na trdnih površinah zaradi delovanja viskoznih sil. Rezultirajoča viskozna sila na nestisljiv delec tekočine je podana z lokalnim vrtinčnim gradienptom. Za tokove tekočin majhne viskoznosti je rezultirajoča viskozna sila pomembna le v točkah tokovnega polja velikih vrtinčnih gradienmov. Enačba prenosa vrtinčnosti (2.17) je močno nelinearna PDE zaradi zmnožka hitrosti \vec{v} in vrtinčnosti $\vec{\omega}$ v konvektivnem in deformacijskem členu, hkrati pa je hitrost kinematično odvisna od vrtinčnosti. Zaradi te vsebovane nelinearnosti je kinetika splošnega viskoznega gibanja, kar pa še posebej velja za tokove z velikimi vrednostmi Reynoldsovega števila, numerično mnogo teže rešljiva v primerjavi s kinematiko. Prenosna enačba vrtinčnosti in energijska enačba sestavljata vezani sistem enačb prek člena vzgonskih sil, kar še dodatno oteži numerično reševanje.

2.4 Tlačni zapis

Pri »penalty« in hitrostno vrtinčnem zapisu smo tlak izločili iz gibalne enačbe kot osnovno spremenljivko, medtem ko se v zapisu osnovnih spremenljivk pojavlja v obliki gradienta in lahko povzroča numerične nestabilnosti.

Z upoštevanjem vektorske enakosti (1.23) izpeljemo hitrostno-vrtinčno-tlačni zapis gibalne enačbe (2.7), npr. v vektorskem zapisu:

$$\frac{\partial \vec{v}}{\partial t} - \vec{v} \times \vec{\omega} = \nu_0 \Delta \vec{v} - \vec{\nabla} h + F \vec{g} \quad (2.19)$$

Equation (2.17) shows, that the rate of change of the vorticity as we follow a fluid particle, given by the Stokes derivation on the left hand side of equation, due to the vortex stretching, viscous diffusion and buoyancy force, is represented by the terms on the right hand side. The diffusion term is exactly analogous to the term in the momentum statement expressing the momentum diffusion. The stretching term is effective whenever the velocity vector is changing along the vortex lines. For a two-dimensional flow, the vorticity vector $\vec{\omega}$ has just one component perpendicular to the plane of the flow, and it can be treated as a scalar quantity ω . The vortex stretching term is identically zero ($\vec{\omega} \cdot \vec{\nabla} \vec{v} = 0$), reducing the vector vorticity equation to a scalar one for the vorticity ω :

where e_{ij} ($i, j = 1, 2$) is the permutation unit symbol ($e_{12} = +1, e_{21} = -1, e_{11} = e_{22} = 0$).

The essential reason for considering the fluid motion in terms of the vorticity distribution is that the vorticity vector $\vec{\omega}$ (\vec{r}, t) is a solenoidal vector, and so it cannot be produced or destroyed in the interior of homogeneous media under normal conditions. It is produced only at the solid boundaries due to viscous effects. The net viscous force on an incompressible fluid particle is given by the local vorticity gradients. For a low viscosity fluid the net viscous force is significant only at the point in the fluid flow of large vorticity gradients. The vorticity transport statement eq. (2.17) is a highly nonlinear PDE due to the products of velocity \vec{v} and vorticity $\vec{\omega}$ in convective and stretching terms, and the velocity is kinematically dependent on vorticity. Because of this inherent nonlinearity, the kinetics of general viscous motion, and what is drastically true for high Reynolds number values flows, represents greater numerical efforts than that considered by the kinematics. Due to the buoyancy force term, the vorticity transport equation is coupled to the energy equation, making the numerical solution procedure even more severe.

2.4 Pressure Formulation

In penalty and in velocity-vorticity formulation, pressure is eliminated from the momentum equation as a primary variable, while in the primitive variables approach, it appears in the gradient form, and as such can cause numerical instabilities.

In view of the vector identity (1.23) the velocity-vorticity-pressure formulation of momentum (2.7) can be obtained, e.g. in a vector notation:

kjer je h — totalni tlak, definiran s $h(\vec{r}, t) = p/\rho_0 - \vec{g} \cdot \vec{r} + v^2/2$. Enačba (2.19) je linearna enačba za neznane tlačne vrednosti, če uporabimo hitrostno-vrtinčni zapis določitve hitrostnega in vrtinčnega polja [21]. Alternativno tlačno predstavitev oblikujemo tako, da poiščemo divergenco enačbe (2.19) [38], kar se kaže v izrazu:

$$\nabla^2 h = \vec{\nabla} \cdot (\vec{v} \times \vec{\omega} + F \vec{g}) \quad (2.20)$$

ki pomeni eliptični problem robnih vrednosti za totalni tlak Dirichletovih in Neumannovih robnih pogojev.

3. INTEGRALSKA PREDSTAVITEV USTALJENE PRENOSNE ENAČBE

Ustaljena difuzivno-konvektivna enačba je pomemben primer parcialnih in diferencialnih enačb opisa prenosnih pojavov v toku tekočine, npr. prenos topotne energije, globalne količine, vrtinčnosti, disperzijskih problemov itn. Zaradi mešanega eliptično-hiperboličnega značaja omenjene parcialne diferencialne enačbe je numerično reševanje prenosnih procesov v tekočinah neprimerno težje kakor v trdninah. To še posebej velja za tokove z velikimi vrednostmi Reyoldsovih oz. Pecletovih števil, ko postane konvekcija dominantna v primerjavi z difuzijo, oziroma ko hiperbolični značaj prevladuje eliptičnost enačbe.

Obravnavajmo splošno stacionarno nelinearno difuzivno-konvektivno enačbo časovno neodvisnega prenosa poljubne skalarne funkcije $u(\vec{r})$ v homogenem izotropnem in nestisljivem mediju v območju rešitve toka Ω , ograjenem z mejo Γ , npr. podano v tenzorskem kartezijevem zapisu:

$$\frac{\partial}{\partial x_j} \left(a_e \frac{\partial u}{\partial x_j} \right) - \frac{\partial v_j u}{\partial x_j} + \frac{I_u}{\rho_0 c_0} = 0 \quad \text{v/in } \Omega \quad (3.1)$$

kjer je $\vec{v}(\vec{r})$ lokalno solenoidalno hitrostno polje. Spremenljivka $u(\vec{r})$ lahko predstavlja, npr. temperaturo pri topotno prenosnih problemih, koncentracijo v disperzijskih procesih, vrtinčnost v dinamiki tekočin itn. in jo bomo označili kot potencial. Dejanska difuzivnost $a_e(\vec{r}, u)$ in izvorni člen $I_u(\vec{r}, u)$ sta poljubni monotoni funkciji kraja in potenciala. Dejansko difuzivnost a_e lahko vedno razdelimo na stalni a_0 in spremenljivi del $a_N(\vec{r}, u)$:

$$a_e = a_0 + a_N(\vec{r}, u) \quad (3.2)$$

To omogoči preoblikovanje (3.1) v obliko:

$$a_0 \frac{\partial^2 u}{\partial x_j \partial x_j} - \frac{\partial v_j u}{\partial x_j} + \frac{\partial}{\partial x_j} \left(a_N \frac{\partial u}{\partial x_j} \right) + \frac{I_u}{\rho_0 c_0} = 0 \quad \text{v/in } \Omega \quad (3.3)$$

Enačba (3.3) predstavlja eliptični problem robnih vrednosti, tako da sklenemo matematični opis prenosnega pojava z robnimi pogoji, npr. Dirichletovimi, Neumanovimi ali Cauchyevimi na delih meje Γ_1, Γ_2 in Γ_3 :

where h is the total pressure head defined by $h(\vec{r}, t) = p/\rho_0 - \vec{g} \cdot \vec{r} + v^2/2$. The (2.19) can be treated as a linear one for unknown pressure values in the case that the velocity-vorticity formulation is used to determine the velocity and vorticity fields [21]. An alternative pressure representation can be formulated by taking the divergence of (2.19) [38], resulting in a expression:

$$\nabla^2 h = \vec{\nabla} \cdot (\vec{v} \times \vec{\omega} + F \vec{g}) \quad (2.20)$$

which represents an elliptic boundary values problem for the total pressure head evaluation, subject to Dirichlet's and Neumann's boundary conditions.

3. INTEGRAL REPRESENTATION OF STEADY TRANSPORT EQUATION

A steady diffusion-conductive equation is an important class of partial differential equations, governing steady transport phenomena in fluid flow, e.g. transfer of heat energy, momentum, vorticity, dispersion problems, etc. Due to the mixed elliptic-hyperbolic character of mentioned PDE, the numerical solution of transport processes in fluids is much more difficult than those in solids. This is specially true for flows characterised with high Reynolds or Pecllet number values, when convection becomes dominant compared with diffusion, or when the hyperbolic character of the equation predominates over the ellipticity of the equation, respectively. Let us consider a general steady state nonlinear diffusion-conductive equation describing the time nondependent transport of an arbitrary scalar function $u(\vec{r})$ in a homogeneous, isotropic and incompressible medium of solution flow domain Ω bounded by the boundary Γ , e.g. given in indicial notation for a right-handed Cartesian coordinate system [19], [28]:

where $\vec{v}(\vec{r})$ is the local solenoidal velocity field. The variable $u(\vec{r})$ can be interpreted, e.g., as a temperature in heat transfer problems, concentration in dispersion processes, vorticity in fluid dynamics problems etc., and will be referred to as a potential. The effective diffusivity $a_e(\vec{r}, u)$ and the source term $I_u(\vec{r}, u)$ are monotonic space and potential dependent functions. The effective diffusivity a_e can be always partitioned into a constant a_0 and a variable part $a_N(\vec{r}, u)$:

This permits rewriting (3.1) as:

The (3.3) represents an elliptic boundary values problem, thus some boundary conditions have also to be specified, to complete the mathematical description of the transport problem, e.g. Dirichlet, Neumann or Cauchy type boundary conditions have to be known on the part of the boundary Γ_1, Γ_2 in Γ_3 :

$$\begin{aligned} u &= \bar{u} && \text{na/on } \Gamma_1 \\ \frac{\partial u}{\partial x_j} n_j &= \frac{\partial \bar{u}}{\partial n} && \text{na/on } \Gamma_2 \\ \frac{\partial u}{\partial x_j} n_j &= \alpha_u (u - u_f) && \text{na/on } \Gamma_3 \end{aligned} \quad (3.4)$$

kjer je α_u — prenosni koeficient med mejo toka tekočine, definirane z normalno enoto \vec{n} in okolico potenciala u_f .

Glede na uporabo različnih osnovnih rešitev lahko oblikujemo številne numerične modele difuzivno-konvektivne enačbe. Za vse te integralne zapise lahko rečemo, da so zelo stabilni, natančni in razmeroma brez pojava umetne difuzivnosti, znanega pri numeričnem reševanju z metodo končnih razlik oziroma končnih elementov. Glavna omejitev samo robne integralne predstavitev je v tem, da obstajajo osnovne rešitve le za parcialno diferencialne enačbe s konstantnimi koeficienti. Če to ni primer, lahko koeficiente vedno razdelimo na stalni in spremenljivi del, ki je obravnavan na način psevdoprostorninskih sil. Glavna pomanjkljivost takšne robno-območne integralne predstavitev je v potrebnici območni diskretizaciji popisa psevdoprostorninskih sil.

3.1 Integralska predstavitev osnovne rešitve Laplaceove enačbe

Obravnavajmo integralno predstavitev nehomogene eliptične parcialno diferencialne enačbe časovno neodvisnega prenosa poljubne skalarne funkcije $u(\vec{r})$ [4], [5]:

$$\frac{\partial^2 u}{\partial x_j \partial x_j} + b = 0 \quad \text{v/in } \Omega \quad (3.5)$$

kjer člen $b(\vec{r}, u)$ pomeni psevdoprostorninske sile. Z uporabo Greenovih teoremov za skalarne funkcije oziroma preprosto z aplikacijo tehnike utežnih ostankov lahko zapišemo naslednji robno-območni integralni stavek:

$$c(\xi)u(\xi) + \int_{\Gamma} u \frac{\partial u^{*E}}{\partial n} d\Gamma = \int_{\Gamma} \frac{\partial u}{\partial n} u^{*E} d\Gamma + \int_{\Omega} bu^{*E} d\Omega \quad (3.6)$$

kjer sta u^{*E} eliptična osnovna rešitev Laplaceove enačbe:

$$\frac{\partial^2 u^{*E}}{\partial x_j \partial x_j} + \delta(\xi, s) = 0 \quad (3.7)$$

oz. sta ξ in s izvorna točka in točka polja, medtem ko sta $\partial u^{*E}/\partial n$ odvod osnovne rešitve normalno na rob in $c(\xi)$ geometrijsko odvisen prosti člen zaradi Cauchyjeve singularnosti na lev strani (3.6). Z izenačitvijo člena psevdoprostorninskih sil s konvekcijo, nelinearno difuzijo in izvornim členom v (3.3):

where α_u is the transfer coefficient between the fluid flow surface defined by the unit normal vector \vec{n} , and the surrounding ambient at the potential u_f .

Different numerical models for the diffusion-convection equation can now be developed based on different fundamental solutions. All of these integral formulations seemed to be very stable, accurate and relatively free from the phenomenon of artificial diffusion, a well known problem in finite difference or finite elements methods of numerical solution. The only major restriction of the boundary integral representation is that fundamental solutions are only available for PDE with constant coefficients. If this is not the case, the coefficients can be always partitioned into constant and variable parts, which are then treated as pseudo-body forces. The main disadvantages of such a boundary-domain integral representation approach, is that domain discretization is required for the pseudo-body forces.

3.1 Integral representation for a fundamental solution of Laplace's equation

Let us first consider an integral representation of a nonhomogeneous elliptic PDE describing the time non dependent transport of an arbitrary scalar function $u(\vec{r})$ [4], [5]:

where $b(\vec{r}, u)$ represents a pseudo body force term. Using Green's theorems for scalar functions, or simply by applying a weighted residual technique one can write the following boundary-domain integral statement:

where u^{*E} is the elliptic fundamental solution of the Laplace's equation, i.e. the solution of:

in which ξ and s are the source and field points, respectively, while $\partial u^{*E}/\partial n$ its derivative in a direction normal to the boundary, while $c(\xi)$ is geometrically dependent free term accounting for the Cauchy type singularity of the integral on the left hand side of (3.6). Equating the pseudo-body force term b by the convection, nonlinear diffusion and source term in (3.3):

$$b = \frac{1}{a_0} \left[-\frac{\partial}{\partial x_j} \left(v_j u - a_N \frac{\partial u}{\partial x_j} \right) + \frac{I_u}{\rho_0 c_0} \right] \quad (3.8)$$

Izpeljemo naslednji integralski stavek difuzivno-konvektivne enačbe:

$$\begin{aligned} c(\xi)u(\xi) + \int_{\Gamma} u \frac{\partial u^*}{\partial n} d\Gamma &= \int_{\Gamma} \frac{a_c}{a_0} \frac{\partial u}{\partial n} u^* d\Gamma - \frac{1}{a_0} \int_{\Gamma} u v_n u^* d\Gamma \\ &+ \frac{1}{a_0} \int_{\Omega} \left[\left(u v_j - a_N \frac{\partial u}{\partial x_j} \right) \frac{\partial u^*}{\partial x_j} + \frac{I_u}{\rho_0 c_0} u^* \right] d\Omega \end{aligned} \quad (3.9)$$

kjer je $v_n = \vec{v} \cdot \vec{n}$ normalna komponenta hitrosti na rob. Enačba (3.9) velja za prostorske ($j = 1, 2, 3$) kakor tudi za ravninske ($j = 1, 2$) tokove pri uporabi ustreznih prostorskih oziroma ravninskih eliptičnih osnovnih rešitev.

Robno-območna integralska predstavitev (3.9) opisuje prenos skalarne funkcije u v integralski oblik na fizikalno ustrezen način. Glede na to je numerična shema, ki izhaja iz diskretne integralne enačbe, zelo stabilna in natančna. Opazimo lahko, da je difuzija čisto robni problem, podan s prvo robnima integraloma, medtem ko tretji robni integral daje rezultirajoči konvektivni tok veličine u prek kontrolne površine, ki seveda ne obstaja v primeru, ko ne obstaja normalna komponenta hitrosti $v_n = 0$. Območni integral se pojavi zaradi območnih konvektivnih učinkov, ne-linearne difuzije in prispevka izvornega člena na razvoj skalarnega polja. Za harmonične vire lahko ta območni integral preoblikujemo v enakovreden robni integral.

3.2 Integralska predstavitev osnovne rešitve modificirane Helmholtzove enačbe

Integralski zapis difuzivno-konvektivne enačbe lahko temelji tudi na modificirani Helmholtzovi nehomogeni parcialno diferencialni enačbi [26]:

$$\frac{\partial^2 u}{\partial x_j \partial x_j} - \beta u + b = 0 \quad \text{v/in } \Omega \quad (3.10)$$

kjer je parameter β pozitivno število. To diferencialno enačbo preoblikujemo v ustrezeno integralsko predstavitev z uporabo tehnike utežnih ostankov, npr. kar se kaže v integralskem stavku:

$$c(\xi)u(\xi) + \int_{\Gamma} u \frac{\partial u^*}{\partial n} d\Gamma = \int_{\Gamma} \frac{\partial u}{\partial n} u^* d\Gamma + \int_{\Omega} b u^* d\Omega \quad (3.11)$$

kjer je u^* osnovna rešitev modificirane Helmholtzove enačbe:

$$\frac{\partial^2 u^*}{\partial x_j \partial x_j} - \beta u^* + \delta(\xi, s) = 0 \quad (3.12)$$

Člen psevdoprostorninskih sil b predstavlja:

$$b = \frac{1}{a_0} \left[-\frac{\partial}{\partial x_j} \left(v_j u - a_N \frac{\partial u}{\partial x_j} \right) + \frac{I_u}{\rho_0 c_0} \right] + \beta u \quad (3.13)$$

the following integral statement can be written for the diffusion-convective equation

$$\int_{\Gamma} u \frac{\partial u^*}{\partial n} d\Gamma = \int_{\Gamma} \frac{a_c}{a_0} \frac{\partial u}{\partial n} u^* d\Gamma - \frac{1}{a_0} \int_{\Gamma} u v_n u^* d\Gamma$$

$$+ \frac{1}{a_0} \int_{\Omega} \left[\left(u v_j - a_N \frac{\partial u}{\partial x_j} \right) \frac{\partial u^*}{\partial x_j} + \frac{I_u}{\rho_0 c_0} u^* \right] d\Omega \quad (3.9)$$

where $v_n = \vec{v} \cdot \vec{n}$ is the normal velocity component to the boundary. The (3.9) is valid both for space ($j = 1, 2, 3$) and for plane ($j = 1, 2$) flow problems, subject to the use of appropriate space or plane elliptic fundamental solutions.

The boundary-domain integral representation (3.9) describes the transport of the scalar function u in an integral form in a physically adequate manner. Because of this, the numerical scheme resulting from the discretised integral equation is very stable and accurate. Note, that the diffusion is a boundary problem only described by the two boundary integrals, while the third boundary integral gives the resulting convective flux of the quantity u across the control surface, and vanishes for a zero normal velocity component, $v_n = 0$. The domain integral is due to the convective domain effects, non-linear diffusion and the contribution of the source term on the development of the scalar field. For an harmonic source this domain integral part can be transformed to an equivalent boundary integral.

3.2 Integral representation for a fundamental solution of a modified Helmholtz's equation

Next, the integral formulation of a diffusion-conductive equation can be based on a modified Helmholtz's nonhomogeneous PDE [26]:

where the parameter β is a positive number. The above differential equation can be transformed into an equivalent integral representation by applying a weighted residual technique, i.e. resulting in the following integral statement:

where u^* is the fundamental solution of the modified Helmholtz's equation, i.e. the solution of:

The pseudo-body force term b stands now for the terms:

tako da zapišemo naslednjo integralsko enačbo:

$$\begin{aligned} c(\xi)u(\xi) + \int_{\Gamma} u \frac{\partial u^*}{\partial n} d\Gamma &= \int_{\Omega} \frac{a_0}{a_0} \frac{\partial u}{\partial n} u^* d\Omega - \frac{1}{a_0} \int_{\Gamma} u v_n u^* d\Gamma \\ &\quad + \frac{1}{a_0} \int_{\Omega} \left[\left(u v_j - a_0 \frac{\partial u}{\partial x_j} \right) \frac{\partial u^*}{\partial x_j} + \left(\frac{I_u}{\rho_0 c_0} + a_0 \beta u \right) u^* \right] d\Omega \end{aligned} \quad (3.14)$$

Pravilna izbira parametra β ima velik vpliv na konvergenco iterativne sheme. Vsaj za samo nelinearne difuzivne probleme je konvergenco zgornje sheme monotona.

3.3 Integralska predstavitev osnovne rešitve difuzivno konvektivne enačbe

Mogoče najprimernejši in stabilni integralski zapis neodvisno od vrednosti Reynoldsovega števila lahko izpeljemo z uporabo osnovne rešitve difuzivno konvektivne parcialno diferencialne enačbe s konstantnimi koeficienti. Splošni ustaljeni transport s kemično reakcijo prvega reda podaja enačba:

$$a_0 \frac{\partial^2 u}{\partial x_j \partial x_j} - \frac{\partial v_j u}{\partial x_j} - k u + b = 0 \quad \text{v/in } \Omega \quad (3.15)$$

kjer je k reakcijska konstanta. Če želimo razviti integralsko enačbo zgornje parcialno diferencialne enačbe, potrebujemo rešitev (3.15). Ker pa ta obstaja le za ustaljeno hitrostno polje, moramo spremenljivi vektor hitrosti $\vec{v}(\vec{r})$ razstaviti na povprečni konstantni vektor in perturbacijski vektor, tako da je:

$$v_j(r_k) = \bar{v}_j + \tilde{v}_j(r_k) \quad (3.16)$$

To omogoča zapis (3.15) kot:

$$a_0 \frac{\partial^2 u}{\partial x_j \partial x_j} - \frac{\partial \bar{v}_j u}{\partial x_j} - k u - \frac{\partial \tilde{v}_j u}{\partial x_j} + b = 0 \quad (3.17)$$

Diferencialni zapis lahko preoblikujemo v ekvivalentni integralski stavek z uporabo tehnike utežnih ostankov ali Greenovimi teoremi za skalarnе funkcije, kar se kaže v naslednjem integralskem zapisu:

$$\begin{aligned} c(\xi)u(\xi) + a_0 \int_{\Gamma} u \frac{\partial u^*}{\partial n} d\Gamma &= a_0 \int_{\Gamma} \frac{\partial u}{\partial n} u^* d\Gamma - \int_{\Gamma} u v_n u^* d\Gamma + \\ &\quad + \int_{\Omega} \left(u \tilde{v}_j \frac{\partial u^*}{\partial x_j} + b u^* \right) d\Omega \end{aligned} \quad (3.18)$$

kjer je $v_n = \bar{v}_n + \tilde{v}_n = \vec{v} \cdot \vec{n}$ in u^* je sedaj glavna rešitev difuzivno konvektivne enačbe s konstantnimi koeficienti:

$$a_0 \frac{\partial^2 u^*}{\partial x_j \partial x_j} + \frac{\partial \bar{v}_j u^*}{\partial x_j} - k u^* + \delta(\xi, s) = 0 \quad (3.19)$$

rendering the following integral equation:

The proper selection of the parameter β has a great influence on the convergency of the iterative scheme. At least for pure nonlinear diffusion problems the convergency of the above scheme proved to be monotonic.

3.3 Integral representation for a fundamental solution of a diffusion-convective equation

Perhaps the most adequate and stable integral formulation, regardless of the Reynold's number values, can be obtained by using the fundamental solution of a diffusion-convective PDE with constant coefficients. The general steady-state transport, including the first order reaction, can be governed by the equation:

where k stands for the reaction constant. In order to developed an integral equation to the above PDE, a fundamental solution of (3.15) is necessary. Since it exists only for the case of constant velocity fields, the variable velocity vector $\vec{v}(\vec{r})$ has to be decomposed into an average constant and perturbation vector, such that:

this permits rewriting (3.15) as:

The above differential formulation can now be transformed into an equivalent integral statement using a weighted residual technique or Green's theorems for scalar functions, resulting in the following integral formulation:

where $v_n = \bar{v}_n + \tilde{v}_n = \vec{v} \cdot \vec{n}$ and u^* is now the fundamental solution of the diffusion-convective eq. with constant coefficients, i.e. the solution of:

Opazimo lahko, da je predznak konvektivnega člena v (3.15) in (3.19) nasproten, ker operator ni sebi priejen (adjungiran). Člen psevdoprostorninskih sil b izenačimo s členi:

$$(3.18) \quad b = \frac{\partial}{\partial x_j} \left(a_N \frac{\partial u}{\partial x_j} \right) + \frac{I_u}{\rho_0 c_0} \quad (3.20)$$

tako da velja naslednji integralski stavek:

$$\begin{aligned} c(\xi)u(\xi) + a_0 \int_{\Gamma} u \frac{\partial u^* c}{\partial n} d\Gamma &= \int_{\Gamma} a_e \frac{\partial u}{\partial n} u^* c d\Gamma - \int_{\Gamma} u v_n u^* c d\Gamma + \\ &+ \int_{\Omega} \left[\left(u \bar{v}_j - a_N \frac{\partial u}{\partial x_j} \right) \frac{\partial u^* c}{\partial x_j} + \frac{I_u}{\rho_0 c_0} u^* c \right] d\Omega \end{aligned} \quad (3.21)$$

V območnem integralu se pojavlja konvekcija samo zaradi perturbacijskega hitrostnega polja, zaradi česar je ta način, kombiniran s tehniko podobmočij, izredno obetaven za numerično reševanje splošnih problemov toka tekočin z velikimi vrednostmi Reynoldsovega števila.

4. INTEGRALSKA PREDSTAVITEV NESTACIONARNE PRENOSNE ENAČBE

Nestacionarna difuzivno konvektivna enačba predstavlja mešan parabolico-hiperboličen tip parcialne diferencialne enačbe, ki podaja časovno odvisne prenosne pojave v toku tekočine. Časovno odvisen prenosni stavek gibalne količine, vrtinosti, temperature, koncentracije itn. lahko formalno prepoznamo kot isti tip nestacionarne difuzivno konvektivne enačbe.

Obravnavajmo splošno časovno odvisno nelinearno difuzivno konvektivno enačbo, ki opisuje nestacionaren transport poljubne skalarne funkcije $u(\vec{r}, t)$ v homogenem izotropnem mediju in definiranem v območju $R = \Omega \times I$, ki pomeni zmnožek območja Ω in časovnega intervala $I(t_0, t)$:

$$\frac{\partial}{\partial x_j} \left(a_e \frac{\partial u}{\partial x_j} \right) - \frac{\partial u}{\partial t} - \frac{\partial v_j u}{\partial x_j} + \frac{I_u}{\rho_0 c_0} = 0 \quad \text{v/in } R \quad (4.1)$$

kjer sta dejanska difuzivnost a_e in izvorni člen $I_u(\vec{r}, u)$ poljubni prostorsko in časovno odvisni funkciji. Z zamenjavo izraza za variacijo dejanske difuzivnosti oblike (3.2), se (4.14) razdeli na linearini in nelinearni del:

$$a_0 \frac{\partial^2 u}{\partial x_j \partial x_j} - \frac{\partial u}{\partial t} - \frac{\partial v_j u}{\partial x_j} + \frac{\partial}{\partial x_j} \left(a_N \frac{\partial u}{\partial x_j} \right) + \frac{I_u}{\rho_0 c_0} = 0 \quad \text{v/in } R \quad (4.2)$$

Enačba (4.2) predstavlja parabolični problem začetnih in robnih vrednosti. Zato moramo poznati nekatere robne in začetne pogoje, da bi lahko zakrožili matematični opis problema. Robni pogoji so predpisani na ograji Γ_1 , Γ_2 in Γ_3 kot:

It can be noted that the sign of convective term is reversed in (3.15) and (3.19), since the operator is not self-adjoint. The pseudo-body force b can now be equated to terms:

rendering the following final integral statement

$$\begin{aligned} c(\xi)u(\xi) + a_0 \int_{\Gamma} u \frac{\partial u^* c}{\partial n} d\Gamma &= \int_{\Gamma} a_e \frac{\partial u}{\partial n} u^* c d\Gamma - \int_{\Gamma} u v_n u^* c d\Gamma + \\ &+ \int_{\Omega} \left[\left(u \bar{v}_j - a_N \frac{\partial u}{\partial x_j} \right) \frac{\partial u^* c}{\partial x_j} + \frac{I_u}{\rho_0 c_0} u^* c \right] d\Omega \end{aligned} \quad (3.21)$$

Note, that in the domain integral, only convection due to the perturbation velocity field exists, making this approach, combined with a sub-structure technique, the most promising one for a numerical solution of general fluid flow problems for high Reynolds number values.

4. INTEGRAL REPRESENTATION OF AN UNSTEADY TRANSPORT EQUATION

An unsteady diffusion-convective equation represents a mixed parabolic-hyperbolic type of partial differential equations, governing time dependent transport phenomena in fluid flow. The time dependent transport statement for momentum, vorticity, temperature, concentration etc., can be recognised to be formally of the same type as the unsteady diffusion-convection equation.

Let us consider a general unsteady state nonlinear diffusion-convection equation describing the time dependent transfer on an arbitrary scalar function $u(\vec{r}, t)$ in a homogeneous and isotropic medium defined in solution domain $R = \Omega \times I$, representing the product of space Ω and time interval $I(t_0, t)$:

where the effective diffusivity $a_e(\vec{r}, u)$ and source term $I_u(\vec{r}, u)$ are some arbitrary space and potential dependent functions. Substituting expression for effective diffusivity variation of a form (3.2), (4.14) can be partitioned into a linear and nonlinear part in the following manner:

The (4.2) represents a parabolic initial-boundary values problem, so some boundary and initial conditions have to be known to complete the mathematical description of the problem, e.g. Dirichlet, Neumann or Cauchy type boundary conditions have to be prescribed on the part of the boundary Γ_1 , Γ_2 and Γ_3 respectively.

$$\begin{array}{llll} u = \bar{u} & \text{na/on } \Gamma_1 & \text{za/for } t > t_0 \\ \frac{\partial u}{\partial x_j} n_j = \frac{\partial \bar{u}}{\partial n} & \text{na/on } \Gamma_2 & \text{za/for } t > t_0 \\ \frac{\partial u}{\partial x_j} n_j = \alpha_u (u - u_f) & \text{na/on } \Gamma_3 & \text{za/for } t > t_0 \end{array} \quad (4.3)$$

medtem ko so začetni pogoji:

$$u = \bar{u}_0 \quad \text{v/in } \Omega \quad \text{pri/at } t = t_0 \quad (4.4)$$

4.1 Integralski zapis z glavno rešitvijo difuzivne enačbe

Obravnavajmo integralski zapis nehomogene parabolične diferencialne enačbe, ki opisuje neustaljeni prenos poljubne skalarne funkcije $u(\vec{r}, t)$:

$$a_0 \frac{\partial^2 u}{\partial x_j \partial x_j} - \frac{\partial u}{\partial t} + b = 0 \quad \text{v/in } R \quad (4.5)$$

kjer $b(\vec{r}, u)$ predstavlja člen prostorninskih sil. Eناčbo (4.5) preoblikujemo v ustrezeno robno območno integralsko enačbo z uporabo metode utežnih ostankov ali Greenovih teoremov skalarnih funkcij, zapišemo za časovni korak $\tau = t_F - t_{F-1}$:

$$c(\xi)u(\xi, t_F) + a_0 \int_{\Gamma} \int_{t_{F-1}}^{t_F} u \frac{\partial u^*}{\partial n} dt d\Gamma = a_0 \int_{\Gamma} \int_{t_{F-1}}^{t_F} \frac{\partial u}{\partial n} u^* dt d\Gamma + \int_{\Omega} \int_{t_{F-1}}^{t_F} bu^* dt d\Omega + \int_{\Omega} u_{F-1} u_{F-1}^* d\Omega \quad (4.6)$$

In je u^* parabolična rešitev difuzivne enačbe:

$$a_0 \frac{\partial^2 u^*}{\partial x_j \partial x_j} + \frac{\partial u^*}{\partial t} + \delta(\xi, s)\delta(t_F, t) = 0 \quad (4.7)$$

In sta (ξ, t_F) in (s, t) — izvorna in območna točka, $\partial u^*/\partial n$ — odvod v smeri normale na rob območja, $c(\xi)$ — geometrijsko odvisen prosti člen, ki se pojavi zaradi singularnosti Cauchyjevega integrala na levi strani enačbe (4.6).

Z izenačitvijo člena prostorninskih sil b s konvekcijo, nelinearno difuzijo in izvornim členom v enačbi (4.2):

$$b = -\frac{\partial}{\partial x_j} \left(v_j u - a_n \frac{\partial u}{\partial x_j} \right) + \frac{I_u}{\rho_0 c_0} \quad (4.8)$$

lahko izpeljemo naslednjo integralsko enačbo:

$$c(\xi)u(\xi, t_F) + a_0 \int_{\Gamma} \int_{t_{F-1}}^{t_F} u \frac{\partial u^*}{\partial n} dt d\Gamma = \int_{\Gamma} \int_{t_{F-1}}^{t_F} a_n \frac{\partial u}{\partial n} u^* dt d\Gamma - \int_{\Gamma} \int_{t_{F-1}}^{t_F} uv_n u^* dt d\Gamma + \int_{\Omega} \int_{t_{F-1}}^{t_F} \left[\left(uv_j - a_n \frac{\partial u}{\partial x_j} \right) \frac{\partial u^*}{\partial x_j} + \frac{I_u}{\rho_0 c_0} u^* \right] dt d\Omega + \int_{\Omega} u_{F-1} u_{F-1}^* d\Omega \quad (4.9)$$

Eناčbo (4.9) lahko uporabimo tako za prostorsko ($j = 1, 2, 3$) kot ravninsko ($j = 1, 2$) tokovno stanje, razlika je le v uporabljenih osnovnih rešitvah. Robno območni integralski stavek (4.9) predstavlja časovno odvisen prenos skalarne

Let us first consider an integral formulation of a nonhomogeneous parabolic PDE governing time dependent transfer of an arbitrary scalar function $u(\vec{r}, t)$:

where $b(\vec{r}, u)$ stands for a pseudo body force term. The (4.5) can be transformed into an equivalent boundary-domain integral equation by applying a weighted residual technique or Green's theorems for the scalar functions, e.g. written in a time incremental form for the time step $\tau = t_F - t_{F-1}$:

where u^* is the parabolic fundamental solution of the diffusion equation, i.e. the solution of:

in which (ξ, t_F) and (s, t) are the source and field points, respectively, while $\partial u^*/\partial n$ its derivative in a direction normal to the boundary, while $c(\xi)$ is the geometrically dependent free term due to the Cauchy type singularity of the integral on the left hand side of (4.6).

Equating the pseudo body force term b with the convection, nonlinear diffusion and source term in (4.2):

$$b = -\frac{\partial}{\partial x_j} \left(v_j u - a_n \frac{\partial u}{\partial x_j} \right) + \frac{I_u}{\rho_0 c_0} \quad (4.8)$$

one can derive the following integral statement:

The (4.9) is adequate both for space ($j = 1, 2, 3$) and for plane ($j = 1, 2$) flow geometry, subject to the use of appropriate space or plane parabolic fundamental solutions. The boundary-domain integral statement (4.9) represents the time dependent

funkcije u v fizikalno ustrezni integralski obliki. Proses difuzije je opisan s prvima dvema robnima integraloma, medtem ko tretji integral pomeni konvektivni tok prek roba, ki pa izgine v primeru $v_n = 0$. Prvi območni integral se pojavi zaradi konvekcije, nelinearne difuzije in izvorov, medtem ko zadnji območni integral daje vpliv začetnih pogojev na razvoj potencialnega polja naslednjega časovnega intervala.

4.2 Integralska predstavitev z glavno rešitvijo modificirane Helmholtzove enačbe in časovna aproksimacija s končnimi razlikami

Vpeljimo aproksimacijo časovnega odvoda z levo nesimetričnimi končnimi razlikami:

$$\frac{\partial u}{\partial t} \cong \frac{u_F - u_{F-1}}{\tau} \quad (4.10)$$

kar omogoča zapis enačbe (4.5) v obliki:

$$\frac{\partial^2 u_F}{\partial x_j \partial x_j} - \beta u_F + b = 0 \quad \text{v/in } R \quad (4.11)$$

kjer je parameter $\beta = 1/a_0 \tau$. Z izenačitvijo člena prostorninskih sil kot:

$$b = \frac{1}{a_0} \left[-\frac{\partial}{\partial x_j} \left(v_j u - a_N \frac{\partial u}{\partial x_j} \right) + \frac{I_u}{\rho_0 c_0} \right] + \beta u_{F-1} \quad (4.12)$$

lahko izpeljemo integralski zapis:

$$c(\xi)u(\xi) + \int_{\Gamma} u \frac{\partial u^{*H}}{\partial n} d\Gamma = \int_{\Gamma} \frac{a_e}{a_0} \frac{\partial u}{\partial n} u^{*H} d\Gamma - \frac{1}{a_0} \int_{\Gamma} u v_n u^{*H} d\Gamma + \frac{1}{a_0} \int_{\Omega} \left[\left(u v_j - a_N \frac{\partial u}{\partial x_j} \right) \frac{\partial u^{*H}}{\partial x_j} + \frac{I_u}{\rho_0 c_0} u^{*H} \right] d\Omega + \beta \int_{\Omega} u_{F-1} u^{*H} d\Omega \quad (4.13)$$

Robno-območni integral (4.13) je formalno enak enačbi (3.14), razen v primeru dodatnega območnega člena, ki opisuje začetne pogoje. Razvita je popolna implicitna shema, vendar pa so uporabne in izpeljive tudi druge sheme, npr. Crank-Nicholsonova.

5. TESTNI PRIMER

5.1 Tok v odprtih kotanjih

Test v odprtih kotanjih je standarden test za preverjanje robno območne integralske sheme kakor tudi drugih sorodnih shem. Kvadratna kotanja vsebuje izotermni viskozni nestisljivi fluid. Namen analize je določitev gibanja fluida zaradi induciranega gibanja na odprttem zgornjem robu. Analiza je bila izvedena za $Re = 1000$ in $Re = 10000$. Rezultati postopka robnih elementov so primerjeni z rešitvijo »benchmark«, ki jo je podal Ghia z metodo končnih elementov in uporabo zelo goste mreže (128×128 vozlišč). Primerjava pokaže zelo dobra ujemanja rezultatov obeh preračunov. Slika 1 prikazuje geometrijsko obliko problema, diskretizirano območje in robne pogoje. Slike 2 in 3 prikazujeta polje, slike 4 in 5 pa grafe tokovne funkcije.

transport of scalar function u in an integral form in a physically justified way. The diffusion process is described by the first two boundary integrals, while the third boundary integral represents the convective flow on the boundary which vanishes for a zero normal velocity component $v_n = 0$. The first domain integral is due to the convection, non-linear diffusion and source, while the last domain integral gives the initial condition effects on the development of the potential field in the next time interval.

4.2 Integral representation for a fundamental solution of a modified Helmholtz equation and finite-difference approximation in time

Let us introduce on the left, a non-symmetric finite difference approximation of the time derivative in (4.9):

which permits rewriting (4.5) as:

where the parameter $\beta = 1/a_0 \tau$. By taking the pseudo body force term b equal to:

the following integral representation can be obtained:

The boundary-domain integral (4.13) is formally identical to (3.14) except for the additional initial conditions domain term. Although the complete implicit scheme is developed, Crank-Nicholson and others, can be simply formulated in the same manner.

5. NUMERICAL EXAMPLE

5.1 Driven Cavity Flow

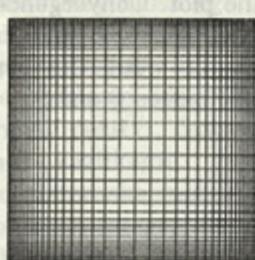
A driven cavity test problem has been used for validating the boundary-domain integral scheme. An isothermal, viscous incompressible fluid is contained in a square cavity. The purpose of the analysis is to determine the flow motion which is induced with steady motion of the top wall. Analysis is performed for high Re number flows: $Re = 1000$ and $Re = 10000$. BEM results are compared with the benchmark solution which is proposed by Ghia, and is obtained by FEM on very fine mesh (128×128 nodes). Comparison shows a very good agreement. The geometry of the problem, discretized model and boundary conditions are presented in Figure 1. Figures 2 and 3 represent the velocity field, while Figures 4 and 5 show

jeva vektora, od tega končno rezultatov na koncu je mogoč. Vrednost povečanja razmerja celic običajno nima vpliva na rezultate, zato je dovoljno za časovno odvisnost rezultata. Če je rezultat nepravilen, potem je potreben na optični način da se preveri ali je rezultat pravilen, kar je v tem primeru nepravilno s končno vektorsko vrednostjo.

Kadar je znano, da je rezultat pravilen, potem se prične s postopkom, ki je predstavljen v robnoj geometriji in robnih pogojih. Če je rezultat nepravilen, potem je potreben ponovno eksperimentišati, kar je v tem primeru nepravilno s končno vektorsko vrednostjo.

Diskretni model: neenakomerna mreža, 41×41 vozlišč, linearna interpolacija za funkcijo, konstantna interpolacija za odvod funkcije, razmerje mreže: $L_{\max}/L_{\min} = 10$, število robnih elementov $N_E = 160$, število notranjih celic $N_C = 1600$, (4 vozliščni četverokotniki), število robnih vozlišč $n = 160$, število robnih in notranjih vozlišč $m = 1681$.

Postopek reševanja: vrtinčno-hitrostna formulacija ($\vec{\omega} - \vec{v}$), impliciten izračun vrednosti v območju, pod-relaksacijska iterativna metoda, reformacija matrike po 5 iteracijah, $r = 0.01$, računalnik: Convex c3-220

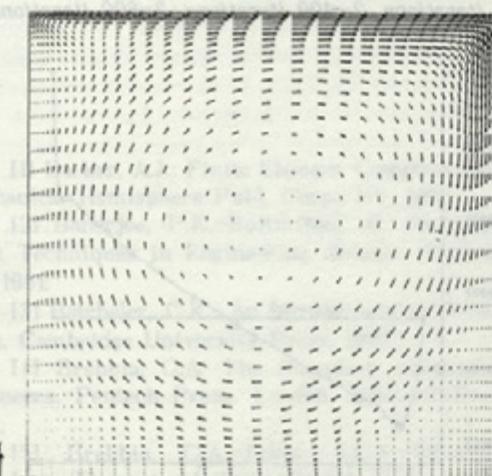


Sl. 1. Mreža, geometrija in robni pogoji.

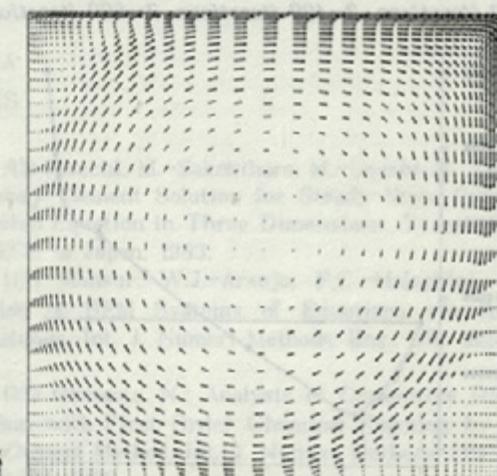
Fig. 1. Mesh, geometry and boundary conditions

Discrete model: Non-uniform mesh, 41×41 nodes, linear interpolation for function, constant interpolation for flux, Mesh gradation: $L_{\max}/L_{\min} = 10$, number of boundary elements $N_E = 160$, number of internal cells $N_C = 1600$, (4 noded quadrilaterals), number of boundary nodes $n = 160$, number of boundary and internal nodes $m = 1681$.

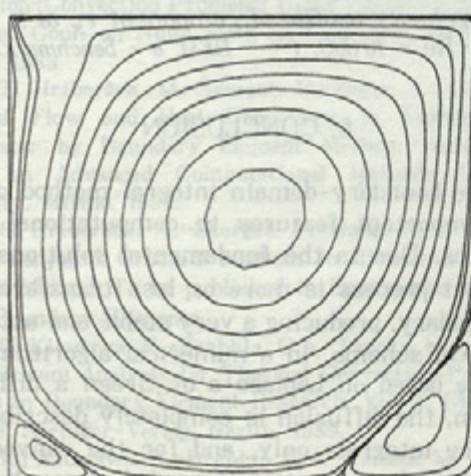
Solution procedure: Vorticity-velocity formulation ($\vec{\omega} - \vec{v}$), implicit computation for domain values, under-relaxation iterative method, matrix reformation after 5 iteration steps, $r = 0.01$, hardware: Convex c3-220



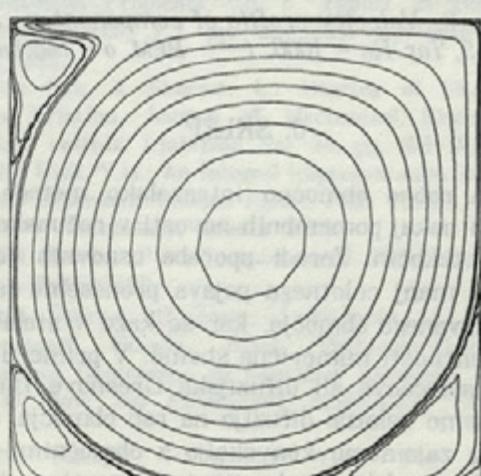
Sl. 2. Hitrostno polje za $Re = 1000$.
Fig. 2. Velocity field for $Re = 1000$.



Sl. 3. Hitrostno polje za $Re = 10000$.
Fig. 3. Velocity field for $Re = 10000$.



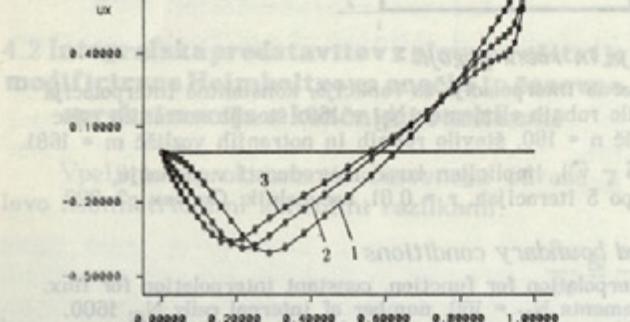
Sl. 4. Tokovne funkcije za $Re = 1000$.
Fig. 4. Streamlines for $Re = 1000$.



Sl. 5. Tokovne funkcije za $Re = 10000$.
Fig. 5. Streamlines for $Re = 10000$.

Slike 6 in 7 podajata konvergenco rešitve, medtem ko je primerjava rezultatov postopek robnih elementov z rešitvami »benchmark« prikazana na slikah 8 in 9.

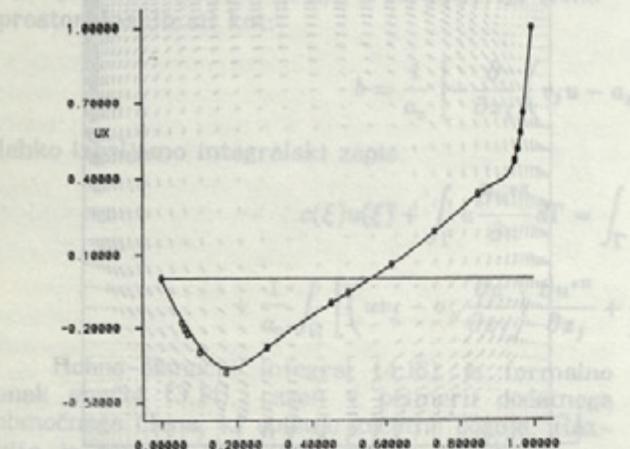
In sliki 6 in 7 opisanih integralih se pojavi zaradi konvergencije nonlinearne difuzijske izvorov, predtem ko znamo, da območni integral daje vpliv na tehtnih pogojev na razvoj potencialnega polja na slednjega časovnega intervala.



Sl. 6. Konvergencia rešitve za $Re = 1000$

(1-200 iteracij, 2-400 iteracij, 3-800 iteracij).

Fig. 6. Convergence to solution for $Re = 1000$
(1-200 iterations, 2-400 iterations, 3-800 iterations).



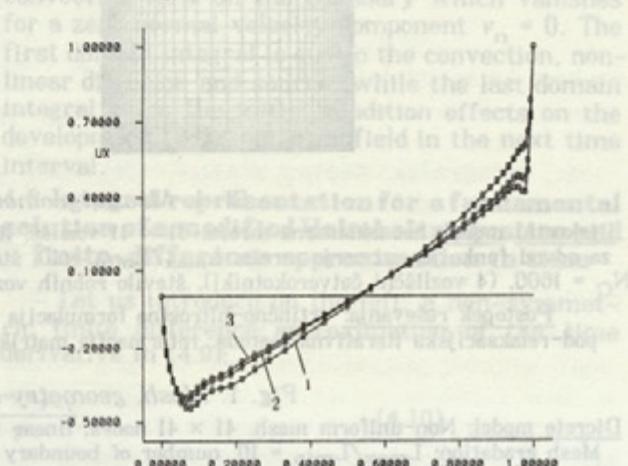
Sl. 8. Hitrostni profil komponente v_x pri $x = 0.5$, za $Re = 1000$, (— - BEM, o - benchmark).

Fig. 8. Velocity profile of component v_x at $x = 0.5$, for $Re = 1000$, (— - BEM, o - benchmark).

6. SKLEP

Za robno območno integralsko metodo smo vpeljali nekaj pomembnih novosti v računske mehanike tekočin. Zaradi uporabe osnovnih rešitev več ali manj celotnega pojava prenesemo na rob obravnavanega območja, kar se kaže v stabilnosti in natančnosti numerične sheme. V primeru uporabe Laplaceove ali difuzijske Greenove funkcije prenesemo celotno difuzijo na rob območja, medtem ko zajarmemo konvekcijo z območnimi integrali. Mnogo bolj učinkovita je numerična shema, ki je zasnovana na difuzijsko konvektivni Greenovi funkciji. Stabilnost numerične rešitve te sheme

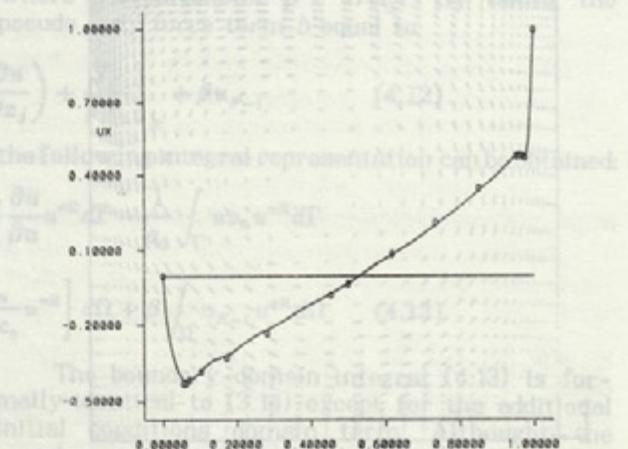
streamline plot. Convergences of the solution are shown on Figures 6 and 7. Comparison of the BEM results with the benchmark solution is shown on Figures 8 and 9.



Sl. 7. Konvergencia rešitve za $Re = 10000$

(1-200 iteracij, 2-400 iteracij, 3-800 iteracij).

Fig. 7. Convergence to solution for $Re = 10000$
(1-200 iterations, 2-400 iterations, 3-800 iterations).



Sl. 9. Hitrostni profil komponente v_x pri $x = 0.5$, za $Re = 10000$, (— - BEM, o - benchmark).

Fig. 9. Velocity profile of component v_x at $x = 0.5$, for $Re = 10000$, (— - BEM, o - benchmark).

6. CONCLUSION

The boundary-domain integral method offers some important features in computational fluid dynamics. Due to the fundamental solutions, the transport process is more or less transferred to the boundary, producing a very stable and accurate numerical scheme. In a numerical algorithm, for example based on Laplace's or Green's diffusion function, the diffusion is completely described by boundary integrals only, and for the convection, the domain discretization is needed. A much more efficient numerical scheme can be formulated regardless of Reynolds number values for the

je neodvisna od velikosti Reynoldsovega števila, saj v območju popišemo le del konvekcije variabilnega dela hitrostnega polja. Nadalje lahko razvijemo za časovno odvisne probleme zapis, ki je zasnovan na eliptični osnovni rešitvi, časovni odvod pa ponazorimo s končnimi razlikami.

Kakor je znano so sistemski matrike, ki se pojavljajo v robno območni integralski tehniki, polno zasedene, zato je uporaba Gaussove neposredne eliminacije nujna. Zaradi tega so potrebe po računalniškem času in spominu velike. Metodo lahko močno izboljšamo z uporabo tehnike podobmočij [7] in mešanih robnih elementov, ki je lahko razvita v ekstremnem primeru tudi do osnutka končnih prostornin. Z uporabo tehnike podobmočij se močno spremeni struktura sistemskih matrik, saj postanejo delno prazne in s tem zelo primerne za uporabo iterativnih metod, kar se lepo kaže pri uporabi metode konjugiranih gradientov, kjer so bili doseženi veliki prihranki računalniškega časa in spomina [8].

diffusion-conductive Green's function, where only the convection for the perturbation velocity field is governed by the domain integrals. A very straightforward formulation of time dependent problems can be developed by using elliptic fundamental solutions and a finite-difference approximation in time.

As is well known, system matrices resulting in boundary-domain technique are completely occupied at least in its original form and the direct solver has to be used, which requires enormous computation times and memory demands. The method can be greatly improved by using a sub-domain technique [7] and mixed-type boundary elements, which can be developed in an extreme case to the concept of finite volume. Using the sub-domain approach, sparsity patterns of the system matrices are considerably improved, and preconditioned conjugate gradient iterative methods can successfully be used with more realistic computation times and memory savings [8].

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