

UDK 531.391:519.61/.64

Vpliv dodatne mase na spremembo lastnih frekvenc**The Influence of Added Mass to a Change of Eigenfrequencies**

MATJAŽ SKRINAR — ANDREJ UMEK

Vsaka sprememba konstrukcije povzroči tudi spremembo njenih dinamičnih karakteristik.

Zato sta opazovanje spremenjanja dinamičnih karakteristik in pravilna razlaga njihovih vzrokov zelo pomembna. Nastanek neugodnih motenj (npr. razpoke) v konstrukciji je treba iz varnostnih razlogov čim hitreje in natančneje zaznati. Inverzno dinamično razpoznavanje je področje dinamike konstrukcij, ki išče vzrok ob znanih posledicah. Za uspešnejšo izvedbo inverznega razpoznavanja je potrebno dobro razumevanje metod dinamičnega razpoznavanja. Prispevek obravnava vpliv dodatne mase na konstrukciji na spremembe lastnih frekvenc obojestransko členkasto podprtga nosilca na tri načine: z analitičnim in dvema numeričnima. Prikazana je tudi primerjava dobljenih rezultatov z rezultati, dobljenimi na podlagi meritev na modelih.

Any change in a structure leads to a change in the dynamic characteristics of the structure, so monitoring changes of the dynamic characteristics and a judgement of their origins is very important. The causes of such disturbing effects (geometrical disorders as, for example, cracks) in a structure have to be detected immediately for reasons of safety. Inverse dynamic identification is a field of dynamics of structures that investigates the reasons of known effects. In order to achieve efficient realisation of inverse identification, we must understand the methods of dynamic identification. This paper deals with the effects of added mass to a change in the eigenfrequencies of a free-supported beam in three ways: with an analytical and two numerical procedures. A comparison of obtained results with data on the real structure is also presented.

0 UVOD

Inverzno dinamično razpoznavanje je področje dinamike konstrukcij, na katerem se rešujejo vprašanja razpoznavanja mehanskih lastnosti konstrukcije po njenih dinamičnih karakteristikah, pridobljenih iz meritev na dejanski konstrukciji. Ker v splošnem dinamične karakteristike temeljnega sistema niso dovolj za enolično razpoznavanje, je treba v računu upoštevati še dinamične karakteristike modificiranega sistema. Možnosti modifikacij je več: od spremembe robnih pogojev, dodačanja ali odvzemanja mas, do spremembe lokalne togosti (dodatne podpore, razpoke). Zamisel inverznega dinamičnega razpoznavanja je neporušna analiza, kar pomeni, da mora biti konstrukcija po izvedbi inverznega razpoznavanja namenjena svojemu prvotnemu namenu. Modifikacija z dodatno maso je tako v splošnem najlaže izvedljiva, kajti dodatno maso lahko po končanem razpoznavanju preprosto odstranimo.

Vpeljava različnih modifikacij seveda različno vpliva na dinamične karakteristike obravnavanega sistema. Pri metodi spremembe robnih pogojev z dodajanjem konstruktivnih elementov (povečanjem togosti) se lastne frekvence zvišujejo, medtem ko se pri dodajanju mase in uvedbi razpok lastne frekvence znižujejo.

0 INTRODUCTION

Inverse dynamic identification is a field of the dynamic of structures in which the problems of identification of the structural parameters are solved on the basis of data obtained from measurements on a real structure. In general, such data is not enough for the unique identification of the structural parameters, so some modifications of the structure must be implemented. Such modifications are: change of the boundary conditions, change of the mass and change of the stiffness of the structure, such as with additional supports. The idea of inverse identification is that the structure is capable of serving to its purpose without any side effects after identification is completed. From this point of view, it is obvious that a change of mass is one of the best solutions.

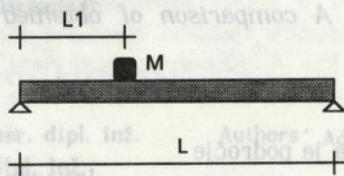
The introduction of different modifications clearly results in different effects on the dynamic characteristics of the system. When the stiffness is varied with additional supports, the eigenfrequencies increase, but adding supplementary mass decreases the eigenfrequencies.

Enačbe gibanja za splošno konstrukcijo so zelo zapletene, tako da si pri računanju odziva največkrat pomagamo z numeričnimi metodami. Ena najbolj znanih in uspešnih je metoda končnih elementov, pri kateri konstrukcijo diskretiziramo z izbranim številom elementov. Dodaten računski napor je potreben, če želimo opazovati spreminjaњje dinamičnih karakteristik glede na različne lokacije dodatne mase. Masno matriko je treba formalno ponovno sestaviti in rešiti enačbe gibanja. Zato je parmetno pri preprostih konstrukcijah (npr. nosilcih in stebrih), obravnavati gibalno enačbo.

Glavna utemeljitev tega prispevka je prikaz preprostih metod za opazovanje sprememb lastnih frekvenc v odvisnosti od lokacije in velikosti dodatne mase. V prispevku je obravnavan obojestransko členasto podprt nosilec, vendar je metodo mogoče zlahka prenesti na druge vrste preprostih konstrukcij. Uspešnost metod je bila preizkušena s primerjavo rezultatov, dobljenih s simuliranjem s končnimi elementi, kakor tudi z meritvami na stvarnem modelu.

1 TEORETIČNI MODEL

Obravnavajmo enostavno podprt nosilec, ki ima na razdalji L_1 od leve podpore dodano maso M (sl. 1).



Sl. 1. Model nosilca

Fig. 1. Model of the beam

Enačbo lastnega nihanja upogibnih nosilcev zapišemo kot:

$$m \frac{\partial^2 v}{\partial t^2} + EI \frac{\partial^2}{\partial x^2}$$

kjer so: m — masa na enoto dolžine, EI — produkt modula elastičnosti E in vztrajnostnega momenta I , v — prečni pomik, za katerega velja $v = v(x, t)$.

Rešitev poiščemo z metodo ločitve spremenljivk v obliki:

$$v(x, t) = u(x) Y(t)$$

Nosilec razdelimo v dva dela z enakimi karakteristikami, kar da dve funkciji u_1 in u_2 za levi in desni del od koncentrirane mase:

$$u_1(\beta) = A_1 \sin(\lambda\beta) + A_2 \cos(\lambda\beta) + A_3 \sinh(\lambda\beta) + A_4 \cosh(\lambda\beta) \quad (3)$$

$$u_2(\beta) = B_1 \sin(\lambda\beta) + B_2 \cos(\lambda\beta) + B_3 \sinh(\lambda\beta) + B_4 \cosh(\lambda\beta)$$

kjer velja:

Equations of motion are very complicated so we have to use numerical methods. One such is the finite element method, whereby the structure is discretised into elements of finite length. If we want to monitor the change of the dynamic characteristics caused by the added mass, we need to solve new equations of motion. In analysing simple structures, as for example beams and columns it is therefore more instructive to consider directly the equation of motion.

The main purpose of this paper is to survey simple methods for monitoring the change of eigenfrequencies in relation to the amplitude and location of the added mass. In this paper, a simple supported beam is considered although the methods can be applied to all other simple structures. The efficiency of the methods has been tested by comparison with the results obtained both with the simulation with finite elements and with measurements on the real structure.

1 THEORETICAL MODEL

Let us consider a simple supported beam that has an additional mass at distance L_1 from the left support (Fig. 1).

The equation of free transverse vibration of beams is written as follows:

$$\left(EI \frac{\partial^2 v}{\partial x^2} \right) = 0 \quad (1)$$

where: m — the mass distribution on the unit of length, EI — the product of the elasticity modulus with moment of inertia, v — lateral displacement, which is $v = v(x, t)$.

We search for the solution of the equation of motion in the form:

$$u(x) Y(t) \quad (2)$$

We devide the beam into two parts with equal characteristics that lead to two functions u_1 and u_2 for the left and right parts of the added mass respectively.

$$\lambda^4 = \frac{\omega^2 m L^4}{EI} \quad \text{in} \quad \beta = \frac{x}{L} \quad (4a, b)$$

in je ω lastna krožna frekvence. Koeficiente A_1 in B_1 izračunamo iz naslednjih robnih pogojev:

and ω represents the circular eigenfrequency.

The coefficients A_1 and B_1 can be computed from the following boundary conditions:

$$u_1(\beta)|_{\beta=0} = 0 \quad u_1''(\beta)|_{\beta=0} = 0 \quad u_2(\beta)|_{\beta=1} = 0 \quad u_2''(\beta)|_{\beta=1} = 0$$

$$u_1(\beta)|_{\beta=R} = u_2(\beta)|_{\beta=R} \quad u_1'(\beta)|_{\beta=R} = u_2'(\beta)|_{\beta=R} \quad u_1''(\beta)|_{\beta=R} = u_2''(\beta)|_{\beta=R}$$

$$EI(u_1'''(\beta) - u_2'''(\beta))|_{\beta=R} + M\omega^2 u_1(\beta)|_{\beta=R} = 0 \quad (5),$$

kjer pomeni $R = L_1/L$.

Z' je označen odvod po x , torej $(') = \frac{d}{dx} (\)$.

where $R = L_1/L$.

The derivate with respect to x is denoted with', i.e.: $(') = \frac{d}{dx} (\)$.

Maso nosilca zapišemo kot $M_N = Lm$.

The mass of the beam is given as $M_N = Lm$. The first two conditions lead to $A_2 = A_4 = 0$ and the remaining six constants have a nontrivial value if the next equation is valid:

$$|\Delta_1| = \frac{M_N}{\lambda} |\Delta_1| + M |\Delta_2| \equiv 0 \quad (6),$$

where:

$$|\Delta_1| = \begin{vmatrix} \sin(\lambda R) & \sin(\lambda R) & -\cosh(\lambda R) & -\sinh(\lambda R) & -\cos(\lambda R) & -\sin(\lambda R) \\ \cosh(\lambda R) & \cos(\lambda R) & -\sinh(\lambda R) & -\cosh(\lambda R) & \sin(\lambda R) & -\cos(\lambda R) \\ \sinh(\lambda R) & -\sin(\lambda R) & -\cosh(\lambda R) & -\sinh(\lambda R) & \cos(\lambda R) & \sin(\lambda R) \\ 0 & 0 & \cosh(\lambda) & \sinh(\lambda) & \cos(\lambda) & \sin(\lambda) \\ 0 & 0 & \cosh(\lambda) & \sinh(\lambda) & -\cos(\lambda) & -\sin(\lambda) \\ \cosh(\lambda R) & -\cos(\lambda R) & -\sinh(\lambda R) & -\cosh(\lambda R) & -\sin(\lambda R) & \cos(\lambda R) \end{vmatrix} \equiv 0$$

in

and

$$|\Delta_2| = \begin{vmatrix} \sinh(\lambda R) & \sin(\lambda R) & -\cosh(\lambda R) & -\sinh(\lambda R) & -\cos(\lambda R) & -\sin(\lambda R) \\ \cosh(\lambda R) & \cos(\lambda R) & -\sinh(\lambda R) & -\cosh(\lambda R) & \sin(\lambda R) & -\cos(\lambda R) \\ \sinh(\lambda R) & -\sin(\lambda R) & -\cosh(\lambda R) & -\sinh(\lambda R) & \cos(\lambda R) & \sin(\lambda R) \\ 0 & 0 & \cosh(\lambda) & \sinh(\lambda) & \cos(\lambda) & \sin(\lambda) \\ 0 & 0 & \cosh(\lambda) & \sinh(\lambda) & -\cos(\lambda) & -\sin(\lambda) \\ \sinh(\lambda R) & \sin(\lambda R) & 0 & 0 & 0 & 0 \end{vmatrix} \equiv 0$$

Ker velja enačba (4a) in so koeficienti EI , m in L dani z geometrijsko obliko prereza in materialom, vidimo, da je enačba (6) samo funkcija lastne krožne frekvence ω , nosilca z dodatno maso. Velja $\omega = \omega_0 - \Delta\omega$, kjer je ω_0 lastna frekvanca izvirnega sistema (brez dodatne mase).

Zapišemo:

Because of eq. (4a) and since the coefficients EI , m and L are given with the geometry of the cross section and with the material, it is clear that eq. (6) is only the function of the eigen circular frequency of the beam with added mass.

If we rewrite the circular eigenfrequency as $\omega = \omega_0 - \Delta\omega$ where ω_0 is the circular eigenfrequency of the original system (without added mass) we can further write:

$$f(\omega) = f(\lambda(\omega)) = f(\omega_0 - \Delta\omega) = \frac{1}{\lambda} \frac{|\Delta_1|}{|\Delta_2|} = -\frac{M}{M_N} \quad (7).$$

Za $M \equiv 0$ dobimo:

$$f(\omega_0) = \frac{1}{\lambda_0} \frac{|\Delta_1(\lambda_0)|}{|\Delta_2(\lambda_0)|} = 0 \quad (8)$$

2 PRIBLIŽNA METODA ZA MAJHNE MASE

Funkcijo $f(\omega)$ (7) razvijemo v Taylorjevo vrsto okoli ω_0 , upoštevamo samo prva dva člena in enačbo (8), kar da:

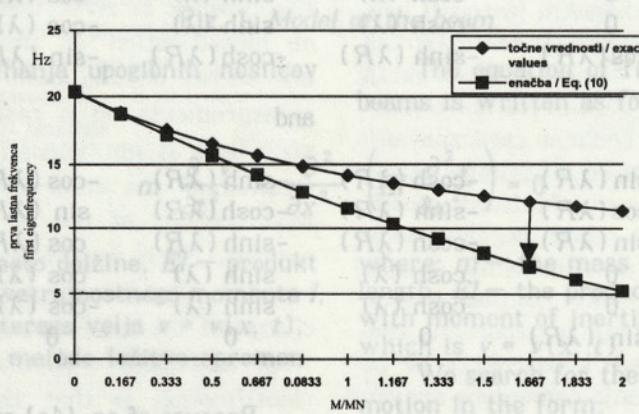
Glavna utemeljitev tega prispevka je prikaz prostih metod za opazovanje sestavljenih frekvenc v odvisnosti od lokacije in datne mase. V prispevku je obravnavan običajni členkoč podprt nastopev. Vendar pa je bilo načrtovano tudi raziskovanje z uporabo elementov končnih razdelitev.

Lastno frekvenco sistema izrazimo kot:

$$\omega = \omega_0 \left(1 + \frac{2M}{M_N} \frac{\partial \lambda}{\partial \omega} \Big|_{\omega_0} \Delta \omega \right) \quad (10)$$

Eqačba (10) pomeni aproksimacijo rešitve v obliki analitičnega izraza, ki je praktično uporaben samo v območju $M \ll M_N$, kar je razvidno s slike 2. Natančnost rešitve, dobljene z uporabo enačbe (10), se namreč zmanjšuje z večanjem dodatne mase M . V primeru, prikazanem na sliki 2, je upoštevana vrednost $M_N = 3,055$ kg.

zapišemo kot:



Sl. 2. Prikaz ujemanja rešitev, dobljenih z uporabo enačbe (10), s točnimi rešitvami

Fig. 2. Comparison of the solutions obtained by equation (10) with exact solutions

Opisani postopek je mogoče modificirati tako, da pri razvoju enačbe (7) v Taylorjevo vrsto upoštevamo tudi člene višjega reda, npr.:

For $M \equiv 0$ we obtain:

$$we are very complicated so we have to use numerical methods. One such is the finite element method, whereby the structure is divided into elements of finite length. In this paper, a simple beam is considered although the method can be applied to all other simple structures. The beam is discretized into four elements of equal length. The mass is distributed over the beam, with the center of gravity located at the midpoint of each element. The mass per unit length is given by the formula: (8).$$

2 APPROXIMATE METHOD FOR SMALL MASSES

If we expand the function $f(\omega)$ into Taylor's series around ω_0 and consider only the first terms and eq. (8) we obtain:

The eigenfrequency is then:

$$the eigenfrequency is then: (9).$$

Eq. (10) presents only an approximation of the solution in an analytical form that can be used only under condition $M \ll M_N$, as demonstrated by figure 2. The accuracy of the solution obtained with the use of eq. (10) decreases with the increase of the added mass M . In the example shown in figure 2, the mass of the beam was $M_N = 3.055$ kg.

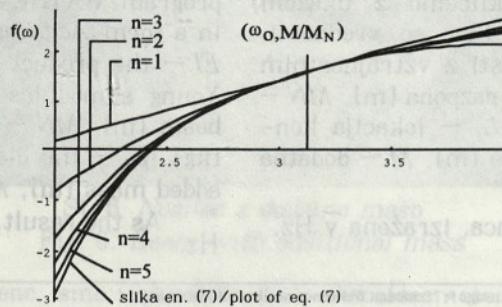
Let us consider a simple supported beam that has a constant cross-section and is made of a material with a constant elasticity modulus. The beam is fixed at both ends and has a length of L . The lateral displacement u is defined as the deflection of the beam from its original position. The equation of motion of the beam is given by the following differential equation: (11).

The described procedure can be further modified. In Taylor's expansion of eq. (7) we consider also higher terms, as for example:

$$f(\omega) = - \frac{\partial f(\omega)}{\partial \omega} \Big|_{\omega_0} \Delta \omega + \frac{\partial^2 f(\omega)}{\partial \omega^2} \Big|_{\omega_0} \frac{\Delta \omega^2}{2!} - \frac{\partial^3 f(\omega)}{\partial \omega^3} \Big|_{\omega_0} \frac{\Delta \omega^3}{3!} + \dots \quad (11).$$

V tem primeru ne dobimo analitične rešitve, temveč moramo enačbo (11) rešiti numerično. Poiskati moramo rešitve polinoma n -tega reda, kjer je n red najvišjega odvoda v aproksimaciji. Uspešnost metode prikazuje slika 3.

*beau ed nadi jeam bns ADITA
erterw AM, d MM d VYzam
ditiw siteni lo inšom edt lo
edt lo dragni edt – A, [m]
meed lantito edt lo zemn edt
edt ot dobla tis tis edt*



Sl. 3. Razvoj funkcije v Taylorjevo vrsto okoli točke $(\omega_0, M/M_N)$ za različne n

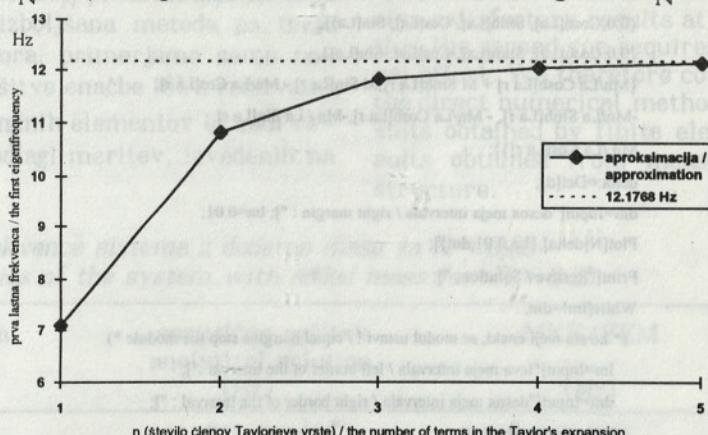
Fig. 3. The expansion of the function in the Taylor's series around the point $(\omega_0, M/M_N)$ for different n

Slike je jasno razvidno, da je prvotna funkcija aproksimirana s Taylorjevo vrsto okoli točke $(\omega_0, M/M_N)$. Rešitev enačbe (11) pomenijo presečišča krivulj z abscisno osjo. Točnost aproksimacije je odvisna od n in razmerja M/M_N . Slika 4 prikazuje konvergenco rezultatov za različne n za vrednosti $M = 5$ kg in $M_N = 3,055$ kg.

*ustreznih rezultatov
veliko računskega
posredne numerične
dobljenimi z metodo
zultati dobavljenimi na
dejanškem nsticu.*

In this case, we no longer obtain an analytical solution and so we must solve eq. (11) numerically. We search for the solution of the polynomial of the n -th order, where n is highest order of the derivative in the approximation. The efficiency of the method is shown in figure 3.

Fig. 3 shows the original function developed in Taylor's series around the point $(\omega_0, M/M_N)$. The solutions of eq. (11) for various values of n are represented as the crossings of the curves with the abscise axe. The accuracy of the approximation depends on n and on the ratio M/M_N . Fig. 4 shows the convergence of the results for different n for values $M = 5$ kg and $M_N = 3.055$ kg.



Sl. 4. Konvergencia rešitve v odvisnosti od števila členov v Taylorjevi vrsti

Fig. 4. The convergence of the solution according to the number of terms in the Taylor's expansion

3 NEPOSREDNA NUMERIČNA REŠITEV ENAČBE (7)

Iz primerjave opisanih postopkov vidimo, da ima vsak svoje prednosti in pomanjkljivosti. Prvi postopek daje analitične rešitve, ki pa so zadovoljive v dokaj ozkem območju. Drugi postopek daje sicer bistveno točnejše rezultate, vendar terja mnogo večji računski napor (izračun višjih odvodov), rešitve pa je mogoče dobiti samo z numeričnimi postopki. Obema postopkom pa je skupna pomanjkljivost, da moramo poznati lastne frekvence glavnega sistema. Zato je pametno poiskati numerične rešitve problema neposredno, torej brez aproksimacij. Podan je modul za program MATHEMATICA, ki po vhodnih podatkih najprej izriše

3 THE DIRECT SOLUTION OF EQUATION (7)

From the comparison of the two methods presented above, we can see that each has advantages and disadvantages. The first procedure gives analytical solutions which are accurate only in a narrow area. The second one gives better results but requires greater effort (computation of a high order derivatives) and the solutions can only be found by numerical procedures. Both solutions have in common the disadvantage that the eigenfrequencies of the original structure must be known. It is therefore straightforward to search for the solutions directly, without any approximations. A module for the program MATHEMATICA is presented that computes the eigenfrequency of a structure with added mass. The program first

sliko enačbe (6) v intervalu, ki ga definira uporabnik, nato pa znotraj podanih intervalov poišče rešitve enačbe (6) po spremenljivki λ in izračuna pripadajoče krožne frekvence.

Modul (sl. 5) najprej vnesemo v program MATHEMATICA, nato pa pokličemo z ukazom *DodatnaMasa[El, L, MN, L₁, M]*, kjer so vrednosti: *El* – produkt modula elastičnosti z vztrajnostnim momentom [Nm²], *L* – dolžina razpona [m], *MN* – masa osnovnega nosilca [kg], *L₁* – lokacija koncentrirane mase od leve podpore [m], *M* – dodatna masa [kg].

Rezultat je lastna frekvencia, izražena v Hz.

plots eq. (6) in the boundaries given by the user, then searches for the solution of eq. (6) with respect to λ , and then computes the appropriate circular frequencies.

The module (fig. 5) must be first read to the program MATHEMATICA and must then be used in a form *AdditionalMass[El, L, MN, L₁, M]*, where *El* – the product of the moment of inertia with Young's modulus [Nm²], *L* – the length of the beam [m], *MN* – the mass of the original beam [kg], *L₁* – the distance from left support to the added mass [m], *M* – added mass [kg].

As the result, we obtain the eigenfrequency in Hz.

```
DodatnaMasa::usage = "Izracuna frekvence nosilca z dodatno maso /  
computes the eigenfrequencies of the beam with added mass"  
  
Begin["Private`"];  
  
DodatnaMasa[El_,L_,MN_,L1_,M_]:=Module[{d,delta,omega,La,r,nic,lm,dm},  
r=L/L;  
d={{Sinh[La r],Sin[La r],-Cosh[La r],-Sinh[La r],-Cos[La r],-Sin[La r]},  
{Cosh[La r],Cos[La r],-Sinh[La r],-Cosh[La r],Sin[La r],-Cos[La r]},  
{Sinh[La r],-Cosh[La r],-Sinh[La r],-Cos[La r],Cos[La r],Sin[La r]},  
{0,0,Cosh[La],Sinh[La],Cos[La],Sin[La]},  
{0,0,Cosh[La],Sinh[La],-Cos[La],-Sin[La]},  
{Mn/La Cosh[La r] + M Sinh[La r],M Sin[La r] - Mn/La Cos[La r],  
-Mn/La Sinh[La r],-Mn/La Cosh[La r],-Mn/La Sin[La r],  
Mn/La Cos[La r]}];  
delta:=Det[d];  
dm=Input["desna meja intervala / right margin : "]; lm=0.01;  
Plot[N[delta],{La,0.01,dm}];  
Print["Resitev / Solutions "];  
While[Im!=dm,  
(* ko sta meji enaki, se modul ustavi ! / equal margins stop the module *)  
Im=Input["leva meja intervala / left border of the interval : "];  
dm=Input["desna meja intervala / right border of the interval : "];  
If[Im==dm, Break[]];  
nic=La/.FindRoot[delta==0,{La,{Im,dm}}];  
omega=N[(nic/L)^2 Sqrt[El/(Mn/L)/2/Pi]];  
Print["L=",nic," --> omega =",omega];  
Plot[delta,{La,Im,dm}];  
];  
End]
```

Sl. 5. Modul za program MATHEMATICA

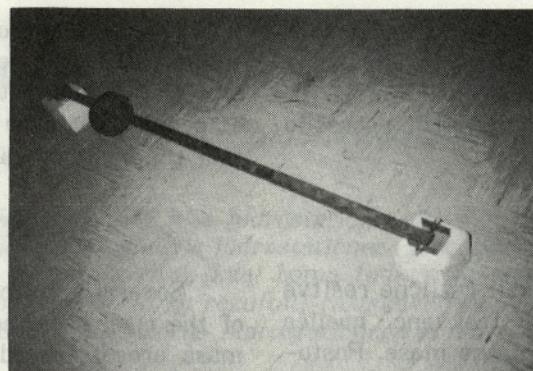
Fig. 5. Module for the program MATHEMATICA

4 NUMERIČNI ZGLED IN PRIMERJAVA REŠITEV

Za potrditev metode je bil analiziran členkasto podprt nosilec (sl. 6) naslednjih karakteristik: $L = 1,8$ m, $m = \rho A = 1,69722$ kg/m³, $M_N = 3,055$ kg, $EI = 2985,36$ Nm², $M = 5$ kg.

4 NUMERICAL EXAMPLE AND A COMPARISON OF THE SOLUTIONS

For a comparison of the solutions, a simple supported beam (Fig. 6) with the following data was analysed: $L = 1.8$ m, $m = \rho A = 1.69722$ kg/m³, $M_N = 3.055$ kg, $EI = 2985.36$ Nm², $M = 5$ kg.



Sl. 6. Nosilec z dodatno maso
Fig. 6. Beam with additional mass

Za izmero lastnih frekvenc smo uporabili akcelerometre. Uporaba teh je primerna zaradi možnosti hitre in preproste namestitve, pomanjkljivost pa je, da vsak akcelerometer doda masi sistema svojo lastno maso. Vendar primerjava mase uporabljenega akcelerometra (54,2 g) in mase nosilca ($M_N = 3,055\text{ kg}$) jasno pokaže, da lahko to maso zanemarimo.

Analizirana sta bila dva primera, in sicer za $R = 0,25$ in $R = 0,5$. Ker smo že prej pokazali, da pri velikem razmerju M/M_N prva metoda ne daje ustreznih rezultatov, izboljšana metoda pa terja veliko računskega napora, primerjamo samo neposredne numerične rešitve enačbe (6) z rešitvami, dobljenimi z metodo končnih elementov in tudi rezultati dobljenimi na podlagi meritev, izvedenih na dejanskem nosilcu.

Preglednica 1: Frekvenca sistema z dodatno maso za $R = 0,25$
Table 1: Frequencies of the system with added mass for $R = 0,25$

lastna frekvenca eigenfrequency	analitična rešitev analytical solution (Hz)	MKE/FEM (Hz)	meritve measurements (Hz)
v_1	12,1768 (7,098*)	12,1951	12,20
v_2	54,6268	54,716	53,0
v_3	163,656	164,561	168,0
v_4	325,330	329,146	—
v_5	443,386	449,574	—

* Rešitev, dobljena z enačbo (10). / * Solution obtained with eq. (10).

Preglednica 2: Frekvenca sistema z dodatno maso za $R = 0,5$
Table 2: Frequencies of the system with added mass for $R = 0,5$

lastna frekvenca eigenfrequency	analitična rešitev analytical solution (Hz)	MKE/FEM (Hz)	meritve measurements (Hz)
v_1	9,79376	9,8078	9,76
v_2	81,332	81,5846	72,0
v_3	135,6305	136,1510	162,3
v_4	325,330	329,1459	—
v_5	420,9525	426,3144	—

Veliko napako pri primerjavi v_3 drugega primera lahko pripisemo meritvam, saj so bile izvedene na robu območja, ki smo ga še lahko izmerili, pa tudi dejstvu, da se nam sistema ni posrečilo ustreznou vzbuditi.

The higher error in the comparison v_3 for the second case can be explained by measurements that were made on the edge of the area which can be considered to be precise, and we can assume that the system was not adequately excited.

MATHEMATICA, nato pa je

Dodatna Masa EI L

EI = produkt modula elastičnosti

V prispevku smo obravnavali različne rešitve problema spremembe lastnih frekvenc nosilca glede na velikost in lokacijo dodatne mase. Postopek smo prikazali na primeru členkasto podprtga nosilca, čeprav ga v splošnem lahko uporabljamo za druge robne pogoje. Karakteristična enačba je bila zapisana simbolično s programskim paketom MATHEMATICA, njene rešitve pa so bile poiskane numerično. Uspešnost postopka je bila preverjena tako z numeričnim simuliranjem z MKE kakor z meritvami na dejanski konstrukciji. Izkazani rezultati so potrdili uspešnost neposredne numerične rešitve tudi pri višjih frekvencah, ki pa z vidika dinamične analize konstrukcij običajno niso več zanimive.

5 SKLEP

5 CONCLUSION

Several solutions for the problem of change of the eigenfrequencies of a beam with an added mass are discussed in the paper. The use was demonstrated on a simply supported beam although it can be easily transferred to similar problems. The characteristic equation was written in symbolic form in the program MATHEMATICA and then solved numerically. The efficiency of the solution was tested both with a numerical simulation by FEM and with measurements on a real structure. The obtained results confirmed the validity of the direct numerical solution also at higher frequencies which are not of primary interest from an engineering point of view.

6 LITERATURA

6 REFERENCES

- [1] Liang, R.Y.-Hu, J.- Choy, F.: Theoretical Study of Crack-Induced Eigenfrequency Changes on Beam Structures. *Journal of Engineering Mechanics*, Vol. 188, 1992/2.
- [2] Gladwell, G.M.L.: *Inverse Problems in Vibration*. Martinus Nijhoff Publisher, 1986.
- [3] Ram, Y.M.-Gladwell, G.M.L.: Constructing a Finite Element Model of a Vibratory Rod from Eigendata. *Journal of Sound and Vibration*, 1993.
- [4] Fajfar, P.: *Dinamika gradbenih konstrukcij*. FAGG, Univerza v Ljubljani, 1984.
- [5] Wolfram, S.: *MATHEMATICA — A System for Doing Mathematics by Computer*. Addison-Wesley Publishing Company, 1992.
- [6] Maeder, R.: *Programming in Mathematica*. Addison-Wesley Publishing Company, 1991.

Naslov avtorjev: mag. Matjaž Skrinar, dipl. inž.

prof. dr. Andrej Umek, dipl. inž.

Fakulteta za gradbeništvo

Univerze v Mariboru

Smetanova 17

62000 Maribor

Avthors's Address: Mag. Matjaž Skrinar, Dipl. Ing.

Prof. Dr. Andrej Umek, Dipl. Ing.

Faculty of Civil Engineering

University of Maribor

Smetanova 17

62000 Maribor, Slovenia

Prejeto: 8.8.1994

Received:

Sprejeto: 21.12.1994

Accepted: